ENS CACHAN, UNIVERSITÉ PARIS SACLAY

# Operads with algebraic structure

M1 MPRI: April-August 2016

Supervised by Marcelo Fiore, Cambridge Computer Laboratory

Mathieu Huot

August 21, 2016

# Summary

This document is a Master 1 internship report for the MPRI (Master Parisien de Recherche en Informatique) supervised by Marcelo Fiore from the Computer Laboratory of Cambridge.

The aim of the internship was first to understand the syntax of the operadic world in an abstract way, *i.e.* in category theory, to have a better understanding of the underlying mechanisms and generalise it to unravel its deep structure. Many results immediately follow from working at a high level of abstraction. It allows one to reproduce clearly many technically involved results in operad theory and makes new connections between Computer Science and Mathematics.

We give a new perspective on the way one can get Leibniz rule for combinatorial species, and show how the chain rule for species can be proved in a more abstract way. We prove a characterisation of operads as $\circ$-monoids and get other new results. We characterize given elements for an operad as coherent maps for the associated $\circ$-monoid. Then we characterize $\circ$-monoids homomorphisms which preserve such maps as operad homomorphism which preserve such elements. We finally characterize operads with operations and equations as initial algebras in a given category. We conclude with initial results and future perspectives as establishing the notion of operads with meta-variables.

This internship was a great opportunity to learn a lot about categorical logic, the theory of substitution, the theory of operads, and to be submerged in current research as well as to attend many research talks of great quality.

# Summary sheet

### General Context

An operad can be seen as a bunch of trees whose nodes are to be seen as operations which have one output going to their parent and a finite number of inputs which corresponds to the number of children, equipped with a composition map to plug these trees together. This operation of plugging trees at the leaves of trees is called grafting. The theory of operads allows us to view mathematical structures as collections of operations, even if the usual description of these structures does not resemble collections of operations at all. For example, the collection of non-intersecting circles inside the unit disc can be viewed as the so-called little discs operad.

Operads are a powerful mathematical tool designed to allow one to treat various algebraic problems uniformly. For instance commutative, associative, and Lie algebras all have their own cohomology theories (Harrison, Hochschild, and Chevalley-Eilenberg respectively). These can all be seen as instances of a single operad cohomology [7]. They also allow one to classify loop spaces and infinite loop spaces [14]. For connected spaces, these are exactly classified as algebras over various operads. Operads also appear in low dimensional topology and topological field theory.

We would like to see the operadic world in a more abstract way to unravel its deep structure, first by understanding its syntax and be able to enrich its structure with operations and equations. We do it in a categoric style, more precisely using universal algebra related to Fiore's work on abstract syntax [5, 2, 4, 3]. We thus base our work on the work of Fiore in abstract syntax along the setting of Joyal's combinatorics species of structure [9]. These species are seen in [16, 11] in a categorical style which we use.

### Problem studied

Mathematicians usually consider algebraic structures in the operadic literature as follows. Choose an algebraic theory on sets (monoids, groups, semirings, rings, modules, vector spaces, *etc.*). Consider operads in the category of algebras for the chosen algebraic theory. Add operations, possibly subjects to equations, to these operads. However the work of Fiore (see [5, 2] for instance) has been doing things differently by essentially permuting the first 2 steps and somehow merging the first and the last one.

We study here how operads can be characterised as $\circ$ monoids in a certain presheaf category. Some of the work we present was folklore. We show how operads can be syntactically characterized using universal algebra and how one can add operations and equations to these operads to reproduce what is done in the operadic world. In particular we show we are able to deal with operations of binding which do not appear in the operadic world whereas they are crucial operations to deal with in Computer Science.

A lot of work has been done in the last two decades to understand higher order syntax for applications such as proof assistants and also to understand algebraic structures in a more abstract way. This process of understanding often unveils structure and this entails deeper understanding, as can been seen in [5, 2]. In this work we aim at a better understanding of operad structure in an abstract world for two main purposes: making links between Computer Science and Mathematics, in particular transport ideas, methodology and techniques from operads to

Computer Science, and also find new applications of the abstract syntax with binding of Fiore in Mathematics.

## Contribution

It was mainly folklore that operads can be seen using universal algebra. This work aims at clarifying it and gives proofs and additional theorems which contribute to unravelling the structure of operads. In particular we show one can see operads in vector spaces abstractly as operads with algebraic structure. One can still add extra structure such as pre-Lie structure on top of that.

The first step was to understand how the subtle grafting process of operads works to be able to understand why a sort of linearity occurs in the process. Then to understand the theory of universal algebra to be able to use it for our purpose. Finally we manage to do so and this approach gives promising results.

## Arguments in favour of its validity

As we are able to fully encompass the operation of grafting from the operadic world, our work seem to be relevant as for its aim. In addition new ideas have shown a way to encompass the operation of substitution at internal nodes, which is a second crucial operation in the operadic world, using results of Fiore [2]. Working abstractly in category theory allows one to get many results for free and provides examples for the theory of Fiore as well as it gives a nice way to work with operads.

## Review and Prospects

Using Joyal's species of structure we manage to characterize syntactically operads in the categorical world as well as we are able to add equations and operations at a high level of modularity. We also give a new proof for the chain rule, a new fact for the Leibniz rule in species and two new characterizations.

The next step would be to be able to encompass the substitution operation of operads within our abstraction. The results of [2] gives us a good idea how to treat this. Our work brings new prospects about applications of equational logic and higher order syntax as well as it helps go further in the understanding of a powerful tool from Mathematics. Being able to fully encompass operads leaves us hope to get new results from tools, such as Koszul duality in operads, to other fields. The next investigation could be about the applications of binding in operads.

# Contents

# Chapter 1

# Introduction

The operad $T$ of non-planar binary trees can be seen as follows. Its element are given by binary trees, where internal nodes are to be thought of as a commutative operator, say $+$. The usual way to define it is as the free operad on a binary operation. However using the notion of operad with algebraic structure, there is another sense in which $T$ can be seen as a free construction. An operad $O$ with algebraic structure $O \cdot O \to O$ is an operad whose above operation is compatible with the composition operation, *i.e.* grafting. An operation $O \cdot O \to O$ is to be thought of as a two argument map. Then the operad of non-planar binary trees is initial in the category of operads $O$ with algebraic structure given by an operation $O \cdot O \to O$.

One can see in this small example that there is a link between free constructions of operads within certain theories, here we just chose a binary commutative operation, and initial objects in categories with algebraic structure. This intuition will guide us throughout this report. We will make the above example formal and generalise it to be able to deal with more sophisticated operads, such as operads in vector spaces, seen as initial objects in some category with algebraic structure. In addition, as in the vector space example, these operations need to satisfy several equations, which should be encompassed by our model.

The relevant high level structure is the notion of monoidal category as we will see. We thus first explain the general context in which we will work and recall many constructions that will help us build the desired structure. Then we build new structures on operads and finally characterize them as initial objects in some category with algebraic operations and equations, as desired. More precisely:

Chapter 2 reviews basic notions of monoidal categories, presheaf categories and some well known facts about them. We then specify to the particular presheaf category of species of structure and finally recall the theory of initial algebras. Chapter 3 reviews the definition of operads, $\circ$-monoids and characterize operads as such monoids. Then we show some new facts on operations on species and some new theorems for the structure of $\circ$-monoids. In chapter 4 we show how operations and equations can be added, characterize syntactically operads with such structure as initial algebras, and give some examples. Chapter 5 sums up new results and show some extensions and possible future work. Some categorical background including coends and most proofs are given in the appendix.

## 1.1 Acknowledgement

I am very grateful to my advisor Marcelo Fiore for introducing me to the topic and giving me good advice and comments. I would also like to thank Ohad Kammar and Philip Saville for many interesting conversations and for their time answering categorical questions. I finally want to say thank you to Jean Goubault-Larrecq for his careful reviewing.

## 1.2 Prerequisites

We assume the reader has a certain familiarity with category theory, in particular with the following: categories, functors, natural transformations, universal properties, dinatural transformations, free structures, monads, limits, colimits, adjunctions, exponentials, Kan extensions, equivalence of categories, the concept of skeletal category. See for example [12, 1] for a good introduction to category theory.

We will be using coends extensively and we assume a certain familiarity with the notion. Appendix 5.3 contains a short summary of several central results that will be used. For more details see in [12, 13].

## 1.3 Notation

$Set$ denotes the category of sets and functions.

$Cat$ denotes the category of small categories and functors.

We denote by $\mathcal{S}_n$ the symmetric group on $n$ elements. Given $\sigma \in \mathcal{S}_n$, $\hat{\sigma}(j_1, \ldots, j_n)$ denotes the permutation of the $n$ blocks $j_i$ as $\sigma$ permutes $n$ elements. Given $\sigma \in \mathcal{S}_n, \tau \in \mathcal{S}_m$, $\sigma \oplus \tau$ denotes the block sum.

We will be mainly working with small categories and size issues will not be addressed here. We use $A \in \mathcal{C}$ to say that $A$ is an object in the category $\mathcal{C}$.

Given an object $c$ in a category $\mathcal{C}$, the neutral arrow is be denoted $1_c$, $id_c$ or even $c$ when the context is clear.

We denote by $[a, b]$ the *homset* of arrows from $a$ to $b$.

We denote by $A \cong B$ the fact that $A$ and $B$ are isomorphic.

When using coends we omit in which category the elements in the sum belong to when the context is clear. We will also often use one integral symbol with the sum taken over several indices, which is unambiguous because of Fubini's theorem. See for example [12].

As we will be working with coends in $Set$, coends are a sum of elements over a quotient and we will denote by $[a]$ the class of equivalence of $a$ in a certain coend.

In particular consider a functor $F : \mathcal{C}^{op} \to Set$ and any class $[a, p, f] \in \int^a F_a \times [c, a]$. $a$ denotes the chosen element in the sum, $p \in F_a$ and $f : c \to a$. We keep this construction for other coends: variables first and then elements in order of appearance in the coend.

We denote with double lines the fact that there is a natural isomorphism, as for example in the case of an adjunction $F \dashv G$:

$$\frac{FA \to B}{\overline{A \to GB}}$$

Similarly when there is a natural transformation we use a single line as in this example:

$$\frac{A + B \to C}{A \to C}$$

Given a strong functor $F$, say over a Cartesian product $\times$, its strength will often be denoted $st_{A,B} : A \times FB \to F(A \times B)$.

$\mathcal{Y}$ denotes the Yoneda embedding. See 2.2.3 for more details.

For a category $\mathcal{C}$, $\widehat{\mathcal{C}}$ denotes $Set^{\mathcal{C}^{op}}$.

An exponential $A^B$ for a symmetric tensor product is denoted by $B \multimap A$.

# Chapter 2

# Background

We first review the general setting in which we will work, *i.e.* in a monoidal category. We will be more interested in a particular monoidal structure on presheaf categories.

## 2.1 Monoidal structure

**Definition 2.1.1.** *A monoidal category is a tuple* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *where* $\mathcal{C}$ *is a category equipped with a bifunctor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*, an object* $I \in \mathcal{C}$ *called the unit, and three natural isomorphisms* $\alpha, \lambda, \rho$ *called the associator, the left unit and the right unit respectively, satisfying the following coherence laws:*

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C} \otimes 1_D"] \arrow[d, "\alpha_{A \otimes B, C, D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A, B \otimes C, D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

*commutes for all* $A, B, C, D \in \mathcal{C}$ *and:*

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

*commutes for all* $A, B \in \mathcal{C}$*.*

We will often omit to give explicitly the three natural isomorphisms given by the definition of a monoidal category.

**Example 2.1.2.** $(Set, \times, 1)$ where $\times$ denotes the Cartesian product and $1$ the terminal object in $Set$, *i.e.* any singleton.

**Definition 2.1.3** (Monoidal functor)**.** *Let* $(\mathcal{C}, \otimes, I_{\mathcal{C}}, \alpha_{\mathcal{C}}, \lambda_{\mathcal{C}}, \rho_{\mathcal{C}})$ *and* $(\mathcal{D}, \bullet, I_{\mathcal{D}}, \alpha_{\mathcal{D}}, \lambda_{\mathcal{D}}, \rho_{\mathcal{D}})$ *be monoidal categories. A monoidal functor* $(F, \phi, \phi_0)$ *is a functor* $F : \mathcal{C} \to \mathcal{D}$ *together with a morphism* $\phi_0 : I_{\mathcal{D}} \to F I_{\mathcal{C}}$ *and a natural isomorphism* $\phi_{A,B} : FA \bullet FB \to F(A \otimes B)$ *satisfying the following coherence axioms, for* $A, B, C \in \mathcal{C}$*:*

$$\begin{tikzcd}
(FA \bullet FB) \bullet FC \arrow[r, "\alpha_{\mathcal{D}}"] \arrow[d, "\phi_{A,B} \bullet 1"'] & FA \bullet (FB \bullet FC) \arrow[d, "1 \bullet \phi_{B,C}"] \\
F(A \otimes B) \bullet FC \arrow[d, "\phi_{A \otimes B, C}"'] & FA \bullet F(B \otimes C) \arrow[d, "\phi_{A, B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[r, "F\alpha_{\mathcal{D}}"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
FA \bullet I_{\mathcal{D}} \arrow[r, "1 \bullet \phi_0"] \arrow[d, "\rho_{\mathcal{D}}"'] & FA \bullet FI_{\mathcal{C}} \arrow[d, "\phi_{A, I_{\mathcal{C}}}"] & I_{\mathcal{D}} \bullet FA \arrow[r, "\phi_0 \bullet 1"] \arrow[d, "\lambda_{\mathcal{D}}"'] & FI_{\mathcal{C}} \bullet FA \arrow[d, "\phi_{I_{\mathcal{C}}, A}"] \\
FA \arrow[r, "F\rho_{\mathcal{C}}"'] & F(A \otimes I_{\mathcal{C}}) & FA \arrow[r, "F\lambda_{\mathcal{C}}"'] & F(I_{\mathcal{C}} \otimes A)
\end{tikzcd}$$

**Example 2.1.4.** For an object $d$ of $Set$, the exponentiation functor $F : (Set^{op}, +) \to (Set, \times)$ sending $a$ to $[a, d]$ is monoidal.

Let $(\mathcal{C}, \star, I, \alpha, \lambda, \rho)$ be a monoidal category.

**Definition 2.1.5** ($\star$-monoid)**.** *Given an element $X \in \mathcal{C}$, a $\star$-monoid is a pair of maps* $I \xrightarrow[e]{} X \xleftarrow[m]{} X \star X$ *such that the following diagrams commute:*

$$\begin{tikzcd}
X \star (X \star X) \arrow[rr, "\alpha"] \arrow[d, "X \star m"'] & & (X \star X) \star X \arrow[d, "m \star X"] & X \star X \arrow[r, "m"] & X \arrow[d, "\lambda"] & X \star X \arrow[r, "m"] & X \\
X \star X \arrow[r, "m"] & X & X \star X \arrow[l, "m"] & I \star X \arrow[u, "e \star X"] \arrow[r, equal] & I \star X & X \star I \arrow[u, "X \star e"] \arrow[r, equal] & X \star I \arrow[u, "\rho"']
\end{tikzcd}$$

**Example 2.1.6.** $(Set, \times, 1)$ is a monoidal category for the Cartesian product $\times$ and the terminal object $1$. Its monoids are the usual ones.

**Definition 2.1.7** ($\star$-monoid homomorphism)**.** *Given two $\star$-monoids* $I \xrightarrow[e]{} X \xleftarrow[m]{} X \star X$ *and* $I \xrightarrow[e']{} Y \xleftarrow[m']{} Y \star Y$ *, an arrow $h : X \to Y$ is said to be a $\star$-monoid homomorphism when the following diagram commute:*

$$\begin{tikzcd}
I \arrow[r, "e"] \arrow[d, equal] & X \arrow[d, "h"] & X \star X \arrow[l, "m"'] \arrow[d, "h \star h"] \\
I \arrow[r, "e'"'] & Y & Y \star Y \arrow[l, "m'"]
\end{tikzcd}$$

*It means $h$ preserves units and multiplications of $\star$-monoids.*

*$\star$-monoids define a category whose objects are $\star$-monoids and morphisms are $\star$-monoid homomorphisms.*

**Example 2.1.8.** Ordinary monoid homomorphisms and monoids in $(Set, \times, 1)$.

## 2.2 Presheaves

**Definition 2.2.1** (Presheaf category)**.** *For a small category $\mathcal{C}$ we define the category $\widehat{\mathcal{C}}$ whose objects are functors $F : \mathcal{C}^{op} \to Set$ and morphisms are natural transformations between them.*

These categories are of crucial importance because they inherit of many nice properties of $Set$ and also of the structure of $\mathcal{C}$. They are used for instance in [5] to deal with abstract syntax with variable binding. The work is done in $\widehat{\mathcal{F}}$ where $\mathcal{F}$ is the category of finite sets and morphisms are all maps. Take an element $X$ of $\widehat{\mathcal{F}}$. Given a finite context $\Gamma \overset{def}{=} \{x_1 \dots x_n\} \in \mathcal{F}$, $t \in X(\Gamma)$ is to be thought as $\Gamma \vdash t : X$. This intuition will later guide us for operads and abstract syntax.

**Proposition 2.2.2.** *Given any small category $\mathcal{C}$, $\widehat{\mathcal{C}}$ is complete and cocomplete with limits and colimits computed point-wise.*

There is an embedding (full and faithful functor) from $\mathcal{C}$ to $\widehat{\mathcal{C}}$ thanks to the following well-known functor:

**Definition 2.2.3** (Yoneda embedding)**.** *Let $\mathcal{C}$ be any locally small category and $\widehat{\mathcal{C}} \overset{def}{=} Set^{\mathcal{C}^{op}}$ the contravariant presheaf category. Let the* Yoneda embedding *be the map given on objects by*

$$\begin{aligned} \mathcal{Y} : \mathcal{C} &\to \widehat{\mathcal{C}} \\ c &\mapsto [-, c] \end{aligned}$$

*and given on arrows by*

$$(f : s \to r) \mapsto [-, f] : [-, s] \to [-, r]$$

**Proposition 2.2.4.** *The Yoneda embedding is well defined, full and faithful.*

The Yoneda embedding transports the structure on $\mathcal{C}$ to $\widehat{\mathcal{C}}$. In particular if $\mathcal{C}$ is (symmetric) monoidal, then $\widehat{\mathcal{C}}$ will be (symmetric) monoidal.

**Theorem 2.2.5** (Day tensor product)**.** *Given a small monoidal category $(\mathcal{C}, \otimes, I)$, there is a unique cocontinuous (which preserve colimits) tensor product on $\widehat{\mathcal{C}}$ which is constructed as the left Kan extension of $\mathcal{Y}(- \otimes =)$ along $\mathcal{Y} \times \mathcal{Y}$, such that the following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd} & \widehat{\mathcal{C}} \\ \mathcal{C} \times \mathcal{C} \arrow[ur, hook, "\mathcal{Y}(-\otimes=)"] \arrow[r, "\mathcal{Y}\times\mathcal{Y}"'] & \widehat{\mathcal{C}} \times \widehat{\mathcal{C}} \arrow[u, dashed, "-\widehat{\otimes}="'] \end{tikzcd}$$

*It is given by the following formula, for all $P, Q \in \widehat{\mathcal{C}}$ and $c \in \mathcal{C}$:*

$$P \widehat{\otimes} Q(c) = \int^{a,b \in \mathcal{C}} P(a) \times Q(b) \times [c, a \otimes b]$$

*The unit of this tensor product is $\mathcal{Y}(I)$. Moreover $\mathcal{Y} : \mathcal{C} \to \widehat{\mathcal{C}}$ is strong monoidal. In particular we have $\mathcal{Y}(c \otimes d) \cong \mathcal{Y}(c) \widehat{\otimes} \mathcal{Y}(d)$ for all $c, d \in \mathcal{C}$.*

One can find an even more general result in [8]. The following general formula will be used several times in the report:

**Theorem 2.2.6** (Density Formula)**.** *Consider a presheaf $P \in \widehat{\mathcal{C}}$. We have the following formula:*

$$P \cong \int^{a} P(a) \times [-, a]$$

There are several proofs of this classical result. We give here a concrete one to give an example of how we work with coends.

*Proof.* We define two natural transformations and show they are inverse to each other.

$$Pc \ni p \overset{\alpha_c}{\mapsto} [c, p, id_c]$$

$$[x, p, f : c \to x] \overset{\beta_c}{\mapsto} Pf(p) \in Pc$$

$\alpha$ is well defined and if $f : d \to c$ then for $p \in Pc$ we have $\alpha_d(P_f(p)) = [d, P_f(p), id_d]$ by definition of $\alpha$ and $\int^a P(a) \times [f, a](\alpha_c(p)) = [c, p, id_c f]$ by definition of the functor $\int^a P(a) \times [-, a]$. By definition of the coend these two classes of equivalence are the same so $\alpha$ is natural.

Take two immediately related representatives of the class $[x, P_g(p), f] = [y, p, gf]$ for $g : x \to y$ and $p \in Py$ ($P$ is contravariant). Then we have $P_f(P_g(p)) = F_{g \circ f}(p) = P_{gf}(p)$ so $\beta$ is well defined. It is also natural.

$\alpha_c \circ \beta_c([x, p, f : c \to x]) = \alpha_c(P_f(p)) = [c, P_f(p), id_c] = [x, p, f]$ by definition of the class of equivalence in the coend.

$\beta_c \circ \alpha_c(p) = \beta_c([x, p, id_x]) = Pid_x(p) = p$ because $P$ is a functor. □

A representable functor is a functor isomorphic to a functor of the form $[-, x]$ for some $x$. This formula is called density because it basically says that every presheaf is a colimit of representable functors, just as every real number is a colimit of rational numbers.

**Lemma 2.2.7** (Yoneda lemma)**.** *Given any (small) category $\mathcal{C}$ we have the following isomorphism in $\widehat{\mathcal{C}}$, natural in both arguments:*

$$[\mathcal{Y}(c), F] \cong Fc$$

*for any $c \in C$ and any presheaf $F \in \widehat{\mathcal{C}}$.*

We can use the Yoneda lemma to compute the exponential in any presheaf category:

**Proposition 2.2.8** (Exponential in a presheaf category)**.** *Let $\mathcal{C}$ be a small monoidal category. Then $\widehat{\mathcal{C}}$ is closed for the Day tensor product. In particular the Day tensor product preserves colimits.*

*Proof.* We only sketch the proof.

If the exponential exists it must satisfy the Yoneda lemma so for $A, B \in \widehat{\mathcal{C}}$ and $c \in \mathcal{C}$ we would have:

$$(A^B)(c) \cong Nat(\mathcal{Y}(c), A^B)$$

Then by the desired adjunction $A \widehat{\otimes} B \dashv A^B$ we would have $Nat(\mathcal{Y}(c), A^B) \cong Nat(\mathcal{Y}(c) \widehat{\otimes} B, A)$. Hence one can define:

$$(A^B)(c) \overset{def}{=} [\mathcal{Y}(c) \widehat{\otimes} B, A]$$

One can check this does satisfy the desired properties for the exponential. □

**Remark 2.2.9.** The Cartesian product we will use is the one in $Set$ because in presheaves limits and colimits are computed point-wise. This product is closed in presheaves and so in particular preserves colimits.

Actually, one can compute more concretely the exponential in presheaves for the particular case where on the the two terms is $\mathcal{Y}(c)$, for $c \in \mathcal{C}$, as the following lemma shows:

**Lemma 2.2.10.** *For all $c \in \mathcal{C}$ we have:*

$$-\widehat{\otimes} \mathcal{Y}(c) \dashv (- \otimes c)^* \overset{def}{=} P \mapsto P(- \otimes c)$$

*Proof.*

$$\begin{aligned} Q^{\mathcal{Y}(c)}(d) &\cong [\mathcal{Y}(d) \widehat{\otimes} \mathcal{Y}(c), Q] \\ &\cong [\mathcal{Y}(d \otimes c), Q] \\ &\cong Q(d \otimes c) \end{aligned}$$

where we used the definition of the exponential in presheaf categories, the fact $\mathcal{Y}$ is strong monoidal and the Yoneda lemma.

Adjoints are defined up to (unique) natural isomorphism so we are done. □

For brevity sake we will from now on write $\cdot$ instead of $\widehat{\otimes}$.

## 2.3 Species

We will now present the category of combinatorial species, a particular presheaf category, introduced by Joyal [9, 10], which he uses to have a new approach of combinatorics using generalised formal power series. It is a generalisation of the idea of generating functions which are for instance used to deal with asymptotic combinatorial problems. This category is equivalent to a smaller one which we use and also call the category of species.

**Definition 2.3.1.** *Let $\mathbb{P}$ be the category whose objects are finite sets and whose functions are bijections.*

*Let $\mathbb{B}$ be the category whose objects are sets $n \overset{def}{=} \{1, \dots, n\}$ for $n \in \mathbb{N}$ and whose functions are permutations. $\mathbb{B}$ is skeletal and there is an equivalence of categories between $\mathbb{P}$ and $\mathbb{B}$. This will help us because $\mathbb{B}$ is a small category (as opposed to $\mathbb{P}$).*

We will sometimes make an abuse of notation by identifying the integer $n$ and the set $n$.

Both $\mathbb{B}$ and $\mathbb{P}$ come equipped with a symmetric monoidal structure $+$ which acts as the disjoint union on objects, the unit being the empty set. More precisely we will see $\mathbb{B}$ as the free symmetric strict monoidal category on $1$, the terminal object in $Cat$. The tensor product $+$ is then given for any objects $n, m \in \mathbb{B}$ by $n + m = \{1, \dots, (n + m)\}$ and for maps $f : n \xrightarrow{\cong} n, g : m \xrightarrow{\cong} m$ by

$$(f + g)(i) = \begin{cases} f(i) & 1 \leqslant i \leqslant n \\ n + g(i - n) & n < i \leqslant n + m \end{cases}$$

We are now able to consider $\widehat{\mathbb{B}}$, the category of species. From the previous section it is closed symmetric monoidal. In addition we have that $\widehat{\mathbb{B}} \simeq \widehat{\mathbb{P}}$ which confirms that it makes sense to call $\widehat{\mathbb{B}}$ the category of species.

**Definition 2.3.2** (Derivative of species)**.** *We define the derivative operation on $\widehat{\mathbb{B}}$ by $\partial \overset{def}{=} \mathcal{Y}(1) \multimap (-)$* i.e. *for all $P \in \widehat{\mathbb{B}}$ and $c \in \mathbb{B}$ we have:*

$$\partial P(c) = P(c + 1)$$

Now recalling the intuition that $t \in P(\Gamma)$ is to be thought as $\Gamma \vdash t : P$, the derivative $\partial$ is to be thought as a context extension as it adds a new variable to a context $\Gamma$. The definition of $+$ ensures that for every such $\Gamma$ it is possible to find a new variable which is not in $\Gamma$. A renaming occurs in De Bruijn style.

There is another (non symmetric) tensor product on $\widehat{\mathbb{B}}$ which encompasses a notion of substitution. This one will encompass the notion of grafting on operads.

**Definition 2.3.3** (substitution tensor product)**.** *We define another tensor product on $\widehat{\mathbb{B}}$ as follows: the bifunctor $\circ : \widehat{\mathbb{B}} \times \widehat{\mathbb{B}} \to \widehat{\mathbb{B}}$, which we call substitution tensor product, is given for $A, B \in \widehat{\mathbb{B}}$ and $d \in \mathbb{B}$ by:*

$$A \circ B(d) \stackrel{def}{=} \int^{m} A_m \times B^{\cdot m}(d)$$

*where $B^{\cdot m} \stackrel{def}{=} B \cdot B \cdot \ldots \cdot B$ is the Day's convolution of $m$ copies of $\mathbb{B}$. The unit is given by $\mathcal{Y}(1)$.*

See for example [14] for further details. As we will work with this tensor product, it is interesting to see what the equivalence classes in the coend look like. They are of the following form $\left[m, p, [x_1, \ldots, x_m, p_1, \ldots, p_m, f : d \to \sum_{i \in m} x_i]\right]$. To put it differently, one chooses an integer $m$ and an element $p \in A_m$. Next one has the Day convolution of $m$ copies of $B$ so by the density formula and Fubini theorem one can see it as a coend with $m$ variables $x_1, \ldots, x_m$. One chooses $m$ elements $p_i \in B_{x_i}$ and finally a bijection $f$ whose codomain is the sum of the $x_i$.

One can check the equality between two classes of equivalence is given, for $\sigma : k \to k', \tau_i : x_i \to y_{\sigma(i)}, p \in P(k'), p_i \in P(x_i), f : d \to \sum_{i \in k} x_i$ ,by

$$\begin{aligned}
&[k, P_\sigma(p), [x_1, \ldots, x_k, p_1, \ldots, p_k, f]] = \\
&[k', p, [y_{\sigma^{-1}(1)}, \ldots, y_{\sigma^{-1}(k')}, P_{\tau_{\sigma^{-1}(1)}}(p_{\sigma^{-1}(1)}), \ldots, P_{\tau_{\sigma^{-1}(k')}}(p_{\sigma^{-1}(k')}), \sigma \square \tau(f)]]
\end{aligned}$$

where $\sigma \square \tau(f)(i, z) = (\sigma(i), \tau_i(z))$, where we used the convenient notation that an element $a$ in a disjoint sum of ordered sets can be identified by the position $z$ in the $i$th set.

**Proposition 2.3.4.** *$- \circ P$ has a right adjoint for every $P \in \widehat{\mathbb{B}}$, making $(\widehat{\mathbb{B}}, \circ, \mathcal{Y}(1))$ closed.*

The coherence laws for $\cdot$ and $\circ$ are recalled in the appendix 5.3.

## 2.4 Initial algebra semantics

**Definition 2.4.1** (Algebra for an endofunctor)**.** *Given an endofunctor $T : \mathcal{C} \to \mathcal{C}$, a $T$-algebra is a pair $(A, \alpha)$ where $A \in \mathcal{C}$ and $\alpha : TA \to A$.*

**Definition 2.4.2** (Algebra homomorphism)**.** *Given an endofunctor $F$ and two $F$-algebras $(A, \alpha), (B, \beta)$, an algebra homomorphism $h : (A, \alpha) \to (B, \beta)$ is a map $h : A \to B$ such that*

$$\begin{tikzcd}
FA \arrow[r, "Fh"] \arrow[d, "\alpha"'] & FB \arrow[d, "\beta"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}$$

*commutes.*

*$F$-algebras form a category whose objects are $F$-algebras and morphisms are $F$-algebras homomorphisms.*

A fix point for an endofunctor $F$ is an algebra over $F$ whose structure map is an isomorphism.

Initial algebras can be seen as least fix points as the following lemmas will show. Under suitable assumptions, these least fix points can be computed inductively as iterated application of $F$ over the initial object of the category just as we would do in a directed-complete partial order (DCPO) for a continuous function.

**Lemma 2.4.3** (Lambek's lemma)**.** *Let $(A, \alpha)$ be an initial $T$-algebra. Then $\alpha$ is an isomorphism.*

The following lemma gives a formal notion of initial algebras as least fixed points, and hence as 'inductive' objects:

**Lemma 2.4.4** (Initial algebras are least fixed points)**.** *Let $(A, \alpha)$ be an initial $T$-algebra. Then $(A, \alpha)$ is a least fixed point of $T$.*

*Proof.* First note that, by Lambek's lemma, $\alpha$ is an isomorphism. Now consider $T(TA \xrightarrow{\alpha} A) = TTA \xrightarrow{T\alpha} TA$. Since $\alpha$ is an isomorphism so is $T\alpha$. But then we get a chain of isomorphisms

$$TTA \cong TA \cong A$$

and so we see that $T(A, \alpha) \cong (A, \alpha)$ and the initial algebra is indeed a fixed point.

$(A, \alpha)$ must be the *least* fixed point by the initiality: if $(B, \beta)$ is also a fixed point, then we have a unique arrow $(A, \alpha) \to (B, \beta)$. □

There is a sufficient condition for initial algebras to be inductive objects.

**Theorem 2.4.5** (Adámek)**.** *Let $\mathcal{C}$ be a category which has colimits of omega-chains and an initial object $0$. Let $F : \mathcal{C} \to \mathcal{C}$ be a functor which preserves $\omega$-chains. Then the structure $\gamma$ of the chain*

$$0 \xrightarrow{i} F(0) \xrightarrow{Fi} \cdots \to F^{(n)}(0) \xrightarrow{F^{(n)}i} F^{(n+1)}(0) \to \ldots$$

*carries a structure of initial $F$-algebra.*

Thus under reasonable assumptions *initial algebras = least fixed points = inductive objects*.

**Example 2.4.6.** Consider the functor $F \stackrel{def}{=} (-)+1 : Set \to Set$ which preserves colimits. $Set$ has all colimits. An initial algebra over $F$ is a pair $(X, \alpha)$ for which we have in particular $X + 1 \cong X$. As we work with sets we can say $X$ is non empty because it has an element $\alpha(1)$ and $X \backslash \{\alpha(1)\}$ is isomorphic to $X$. It reminds us of the induction principle and one can check that $X$ is actually isomorphic to $\mathbb{N}$, the set of natural numbers.

**Definition 2.4.7** (binding signature)**.** *A binding signature $\Sigma = (O, a)$ is a pair consisting of a set of operations $O$ and an arity function $a : O \to \mathbb{N}^*$ [5]. An operator of arity $\langle n_1, \ldots, n_k \rangle$ has $k$ arguments and the $i$-th argument binds $n_i$ variables.*

**Definition 2.4.8.** *The endofunctor functor canonically associated to a signature $\Sigma = (O, a)$ is the functor $\Sigma : \mathcal{C} \to \mathcal{C}$ defined for all object $X$ as:*

$$\Sigma X \stackrel{def}{=} \coprod_{\substack{o \in O \\ a(o) = \langle n_i \rangle_{1 \leqslant k}}} (\partial^{n_1} X) \cdot (\partial^{n_2} X) \cdot \ldots \cdot (\partial^{n_k} X)$$

In our work, $F$ will be the endofunctor associated to a signature which provides chosen operations. These initial algebras are to be thought of as a syntactic construction, computed by induction on terms. These terms are seen as trees whose nodes are operations given by $F$. The associated category of $F$-algebras is to be seen as the category of models in which the map $\beta$ of an algebra $(B, \beta)$ is to be seen as an interpretation of the operations given by $F$. The semantics to a given interpretation of the operations given by $F$ is then the unique map given by initiality.

$$\begin{tikzcd} FA \arrow[r, "F[\![-]\!]"] \arrow[d, "\alpha"'] & FB \arrow[d, "\beta"] \\ A \arrow[r, "[\![-]\!]"'] & B \end{tikzcd}$$

The operation of interest is grafting, *i.e.* a substitution process which occurs at the leaves of terms. This will be encompassed by the action of a monoid. To be able to realize concretely this intuition, we will need the following:

**Definition 2.4.9** (right tensorial strength)**.** *Given a monoidal category* $(\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$ *and a functor* $F : \mathcal{C} \to \mathcal{C}$*, a tensorial strength is a natural transformation:*

$$v \otimes F(w) \overset{\beta_{v,w}}{\to} F(v \otimes w)$$

*such that the following diagrams commute for all* $u, v, w \in \mathcal{C}$*:*

$$\begin{tikzcd} I \otimes F(v) \arrow[r, "\beta_{I,v}"] \arrow[dr, "\lambda_{F(v)}"'] & F(I \otimes v) \arrow[d, "F(\lambda_v)"] \\ & F(v) \end{tikzcd} \qquad \begin{tikzcd} (u \otimes v) \otimes F(w) \arrow[rr, "\beta_{u \otimes v, w}"] \arrow[d, "\alpha_{u,v,w}"'] & & F((u \otimes v) \otimes w) \arrow[d, "F(\alpha_{u,v,w})"] \\ u \otimes (v \otimes F(w)) \arrow[r, "1_u \otimes \beta_{v,w}"'] & u \otimes F(v \otimes w) \arrow[r, "\beta_{u, v \otimes w}"'] & F(u \otimes v \otimes w) \end{tikzcd}$$

A strong endofunctor is a pair given by a functor and a tensorial strength for this functor. One can similarly define a left tensorial strength. Strengths are of importance here because of the following fact. Take a strong endofunctor $(\Sigma, \beta)$ and an initial $\Sigma$-algebra $(X, \alpha)$. Denote by $\sigma$ an operation of substitution $\sigma : X \circ X \to X$ (in our case in will be applied in species). Then the following diagram commutes by definition:

$$\begin{tikzcd} FX \circ X \arrow[r, "\beta_{X, X \circ X}"] \arrow[d, "\alpha \circ 1_X"', "\wr"] & F(X \circ X) \arrow[r, "F\sigma"] & FX \arrow[d, "\alpha", "\wr"'] \\ X \circ X \arrow[rr, "\sigma"'] & & X \end{tikzcd}$$

One can make a category regrouping the notions of operations and substitution [5] as the definition below shows:

**Definition 2.4.10** ($F$-monoid)**.** *Let* $F$ *be a strong endofunctor on a monoidal closed category* $\mathcal{C} \overset{def}{=} (\mathcal{C}, \otimes, I)$*. An* $F$*-monoid* $X \overset{def}{=} (X, m, e, h)$ *consists of a monoid* $(X, m, e)$ *in* $\mathcal{C}$ *and an* $F$*-algebra* $(X, h)$ *such that the following diagram commutes:*

$$\begin{tikzcd} FX \otimes X \arrow[r, "st_{X \otimes X}"] \arrow[d, "h \otimes 1_X"'] & F(X \otimes X) \arrow[r, "Fm"] & FX \arrow[d, "h"] \\ X \otimes X \arrow[rr, "m"'] & & X \end{tikzcd}$$

$F$*-monoids form a category, with morphisms defined as morphisms of* $\mathcal{C}$ *which are both* $F$*-algebras and monoid homomorphisms.*

Now we read the commutativity of the diagram in more concrete terms:

$$f(t_1, \ldots, t_n)[t'/y] = f(t_1[t'/y], \ldots, t_n[t'/y])$$

where $f$ is an operator in the signature $\Sigma = (O, a)$ associated to the functor $\Sigma$, $y$ is a free variable in $t \overset{def}{=} o(t_1, \ldots, t_n)$ and $t', t_1, \ldots, t_n$ are elements of $X$. $[t'/y]$ denotes the substitution by $t'$ of any occurrence of $y$ in $t$.

This means the strength allows us to commute operators $f \in O$ and the substitution operation with respect to all established syntactic rules. More precisely the strength combined with the initiality of the algebra will actually allow us to compute inductively a substitution by case analysis on the term in which we want to substitute a variable. In our case variables will be at the leaves to encompass grafting, as said above.

# Chapter 3

# Operads

## 3.1 Definition

The original definition of operads in topological spaces by May can be found in [14]. Mathematicians usually consider operads set-theoretically enriched with structure such as a topology. As we want to work with general operads and then be able to add structure, it thus makes sense to work over $Set$ and then add equations/operations.

**Definition 3.1.1** (Symmetric operad)**.** *An operad $(\{X(j)\}_{j\in\mathbb{N}}, I, \gamma, \cdot)$ is a family of sets $\{X(j)\}_{j\in\mathbb{N}}$, an element $I \in X(1)$ called the identity, a right action $\cdot$ of the symmetric group $\mathcal{S}_n$ on $X(n)$ for every natural number $n$ and a composition map $\gamma$ such that for all natural numbers $n, k_1, \ldots, k_n$:*

$$\gamma : \begin{cases} X(n) \times X(k_1) \times \cdots \times X(k_n) & \rightarrow \quad X(k_1 + \cdots + k_n) \\ (p, p_1, \ldots, p_n) & \mapsto \quad \gamma(p; p_1, \ldots, p_n) \end{cases}$$

*satisfying the following axioms:*

1. *for all natural numbers $n, j^1_1, \ldots, j^1_{k_1}, \ldots, j^n_{k_n}$, $p^k_i \in X(j^k_i)$ and $j_i \stackrel{def}{=} \sum_{1\leqslant k\leqslant k_i} j^i_k$, $p_i \in X(j_i)$:*

   $\gamma\big(p; \gamma(p_1; p^1_1 \ldots p^1_{k_1}), \ldots, \gamma(p_n; p^n_1 \ldots p^n_{k_n})\big) = \gamma\big(\gamma(p; p_1 \ldots p_n); p^1_1 \ldots p^n_{k_n}\big)$ *(associativity)*

2. *for any $k$ and $p \in X(k)$:*

   - $\gamma(I, p) = p$
   - $\gamma(p, I^k) = p$

   *$I^k$ being $k$ copies of $I$*

3. *for all natural numbers $n, j_1, \ldots, j_n, p \in X(n), p_i \in X(j_i)$ where $j \stackrel{def}{=} \sum_{1\leqslant i\leqslant k} j_i$:*

   - $\gamma(p \cdot \sigma; p_1 \ldots p_k) = \gamma\big(p; p_{\sigma^{-1}(1)}, \ldots, p_{\sigma^{-1}(k)}\big) \cdot \hat{\sigma}(j_1 \ldots j_k)$ *where* $\hat{\sigma}(j_1 \ldots j_k) \in \mathcal{S}_j$
   - $\gamma(p; p_1 \cdot \tau_1 \ldots p_k \cdot \tau_k) = \gamma(p; p_1 \ldots p_k) \cdot (\tau_1 \oplus \cdots \oplus \tau_k)$ *for* $\tau_i \in \mathcal{S}_{j_i}$

**Definition 3.1.2** (Operad homomorphism)**.** *Given two operads $(\{X(j)\}_{j\in\mathbb{N}}, I, \gamma, \cdot)$ and $(\{Y(j)\}_{j\in\mathbb{N}}, I', \gamma', \cdot')$, a natural transformation $h : X \rightarrow Y$ is an operad homomorphism when:*

1. $h_1(I) = I'$
2. $h(\gamma(p; p_1, \ldots, p_n)) = \gamma'(h(p), h(p_1), \ldots, h(p_n))$

3. $h(p \cdot \sigma) = h(p) \cdot' \sigma$

Operads form a category $\mathcal{O}$ whose objects are operads and morphisms are operad homomorphisms.

**Example 3.1.3.** 1. $O(n) \overset{def}{=} \varnothing$ for all $n \neq 1$ and $O(1) \overset{def}{=} \{I\}$ is the initial operad

2. Given a set $X$, setting $O(n) \overset{def}{=} \{f : X^n \to X\}$, $\gamma$ given by the composition of maps and the universal property of the Cartesian product gives rise to a symmetric operad

3. An associative symmetric binary operation, say $+$, which is unital naturally gives rise to an operad

## 3.2 Constructions for operads

The category of operads gives no information about the concrete syntactical structure of operads. This is why we would like to see them equivalently in a rich and known setting. This is exactly what the following theorem does, seeing operads as monoids over the presheaf category $\widehat{\mathbb{B}}$.

For this section $\alpha, \lambda, \rho$ will denote the coherence isomorphisms for $\circ$ as tensor product on $\widehat{\mathbb{B}}$.

**Lemma 3.2.1.** *The category of operads has a forgetful functor $U : \mathcal{O} \to \widehat{\mathbb{B}}$.*

*Proof.* $X$ already has an action on objects of $\mathbb{B}$. Define its action on morphisms as follows, for every integer $n$, $\sigma : n \xrightarrow{\cong} n$ and $p \in X(n)$:

$$X_\sigma(p) \overset{def}{=} p \cdot \sigma$$

Then for $\tau : n \xrightarrow{\cong} n$:

$$X_{\sigma\tau}(p) \overset{def}{=} p \cdot (\sigma\tau) = (p \cdot \sigma) \cdot \tau \overset{def}{=} X_\tau(X_\sigma(p))$$

so $X$ is indeed a functor $\mathbb{B}^{op} \to Set$. The forgetful functor sends $(\{X(j)\}_{j \in \mathbb{N}}, I, \gamma, \cdot)$ to $X \in \widehat{\mathbb{B}}$. □

**Theorem 3.2.2.** *Let $U : \mathcal{M} \to \widehat{\mathbb{B}}$ and $V : \mathcal{O} \to \widehat{\mathbb{B}}$ be forgetful functors. There is an isomorphism of categories between the category $\mathcal{O}$ of operads and the category $\mathcal{M}$ of $\circ$-monoids. In addition it makes the following diagram commute:*

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\cong"] \arrow[dr, "U"'] & & \mathcal{O} \arrow[dl, "V"] \\ & \widehat{\mathbb{B}} & \end{tikzcd}$$

*Proof.* See B. □

Now we can work more abstractly using the structure of presheaf category of species. In particular we now present new theorems dealing with the structure on species. This structure is thus directly linked with operads as we will see.

We will work in $\widehat{\mathbb{B}}$ for the rest of the section.

We sum up the operations we have so far in species:

1. the coproduct $+$
2. the Cartesian product $\times$
3. the Day tensor product$\cdot$
4. the derivative $\partial$
5. the substitution tensor product $\circ$

Each of them will allow us to encompass certain properties which we'll use in the next chapter. We will now see how these operations behave one with another.

### 3.2.1 Leibniz rule

One can define a similar notion than the usual Leibniz rule in polynomials which we call the same. We denote by $\partial$ the functor $\mathcal{Y}(1) \multimap (-)$.

**Definition 3.2.3** (Leibniz rule)**.** *When there is an isomorphism $\partial(A \cdot B) \xrightarrow{\cong} \partial(A) \cdot B + A \cdot \partial(B)$ natural in $A$ ad $B$, one say that Leibniz rule holds.*

If we interpret in more concrete terms, it means that given a variable $x$, a binary operation $o$ and two terms $t_1, t_2$ we have that if $x \in FV(o(t_1, t_2))$, then either $x \in FV(t_1)$ or $x \in FV(t_2)$, but not both, $FV(t)$ denoting the set of free variables of $t$. Another way to phrase it is that the function $FV$ is linear on a binary operation. Finally, a last interesting point of view on this rule is the following. Consider a binary operator $o$ and a context $\Gamma$. Then one can consider two deduction rules:

$$\dfrac{\Gamma, x \vdash t_1 \qquad \Gamma \vdash t_2}{\Gamma, x \vdash o(t_1, t_2)}\ o \text{ left}$$

$$\dfrac{\Gamma \vdash t_1 \qquad \Gamma, x \vdash t_2}{\Gamma, x \vdash o(t_1, t_2)}\ o \text{ right}$$

Then the above isomorphism ensures than if you have $\Gamma, x \vdash o(t_1, t_2)$, then it must come from one of the two above rules.

**Lemma 3.2.4.** *Leibniz rule holds in species.*

*Proof.* We sketch the proof. One can define two natural transformations $\partial(A \cdot B) \xrightarrow{\beta} \partial(A) \cdot B + A \cdot \partial(B)$, $\partial(A) \cdot B + A \cdot \partial(B) \xrightarrow{\alpha} \partial(A \cdot B)$ which they are inverse to each other.

$$[x, y, p, q, f : d \to x + y] \overset{\alpha_d}{\mapsto} [x + 1, y, p, q, \widetilde{f} : d + 1 \to x + 1 + y]$$

where $i \in 1$ is sent by $\widetilde{f}$ to the same position in $1$ in $x + 1 + y$, $i$ so that $f(i) \in x$ is sent to $f(i)$, $i$ so that $f(i) \in y$ is sent to $1 + f(i)$. Notice one could replace $1$ by any integer $c$. $\alpha$ is natural. For the other way one defines

$$[x, y, p, q, f : d + 1 \to x + y] \overset{\beta_d}{\mapsto} [x + 1, y, p, q, \widehat{f} : d \to x - 1 + y]$$

if f sends $1$ to $x$ and $[x, y + 1, p, q, \widehat{f} : x + y - 1]$ otherwise, where $\widehat{f}$ sends $i$ to $f(i)$ if $f(i) < f(1)$ and to $f(i) - 1$ otherwise. $\beta$ is natural. One can check these maps are inverse to each other. Notice the fact that we really use that fact $1$ has a unique element which can be sent either to $x$ or to $y$. $\square$

More generally we have the following result.

**Lemma 3.2.5.** *Given a small symmetric closed monoidal category* $(\mathcal{C}, \cdot, I, \multimap)$ *there always exists a canonical map* $(A \multimap B) \cdot C + B \cdot (A \multimap C) \to (A \multimap (B \cdot C))$, *for* $A, B, C \in \mathcal{C}$.

*Proof.* The evaluation map $t_A : (X \multimap A) \cdot X \to A$ given as the counit of the adjunction $- \cdot Y \dashv Y \multimap (-)$ gives a canonical map as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\big((X \multimap Y) \cdot X\big) \cdot Z \to Y \cdot Z \qquad Y \cdot \big((X \multimap Z) \cdot X\big) \to Y \cdot Z}{\big((X \multimap Y) \cdot X\big) \cdot Z + Y \cdot \big((X \multimap Z) \cdot X\big) \to Y \cdot Z} \text{ univ. prop. of } +}{(X \multimap Y) \cdot (X \cdot Z) + \big(Y \cdot (X \multimap Z)\big) \cdot X \to Y \cdot Z} \text{ associativity of } \cdot}{(X \multimap Y) \cdot (Z \cdot X) + \big(Y \cdot (X \multimap Z)\big) \cdot X \to Y \cdot Z} \text{ symmetry of } \cdot}{\big((X \multimap Y) \cdot Z\big) \cdot X + \big(Y \cdot (X \multimap Z)\big) \cdot X \to Y \cdot Z} \text{ associativity of } \cdot}{\Big(\big((X \multimap Y) \cdot Z\big) + \big(Y \cdot (X \multimap Z)\big)\Big) \cdot X \to Y \cdot Z} \text{ left adjoint pres. colim}}{\big((X \multimap Y) \cdot Z\big) + \big(Y \cdot (X \multimap Z)\big) \to \big(X \multimap (Y \cdot Z)\big)} \text{ adjunction}$$

□

In general the map given by the previous lemma won't be an isomorphism but this gives a canonical map in species, and one can consider the particular case where $X = \mathcal{Y}(1)$. We have the new result that this canonical map is an isomorphism in $\widehat{\mathbb{B}}$. This result is stronger than the only existence of an isomorphism which makes the Leibniz rule holds.

**Theorem 3.2.6** (Leibniz rule)**.** *Given two species* $A$ *and* $B$, *the canonical map* $\partial(A) \cdot B + A \cdot \partial(B) \to \partial(A \cdot B)$ *is an isomorphism.*

*Proof.* See B □

### 3.2.2 Chain rule

Similarly to the Leibniz rule, we may consider the chain rule in species $\partial(F \circ G) \cong (\partial F \circ G) \cdot \partial G$. It is simple calculation to show that this rule holds in species. However, work has been done to give a new and more abstract proof of this result which thus shows the rule will hold for more general categories than species.

The proof uses universal properties and the two specific lemmas below.

**Lemma 3.2.7.** *In* $\widehat{\mathbb{B}}$ *we have the following two properties:*

1. *for all* $m, h \in \mathbb{B}$*:*
$$[m+1, h] \cong \sum_{j \in h} [m, h \backslash \{j\}]$$

2. *for all* $n, a, b \in \mathbb{B}$*:*
$$[n+1, a+b] \cong \int^{q \in \mathbb{B}} [q+1, a] \times [n, q+b] + \int^{q \in \mathbb{B}} [q+1, b] \times [n, a+q]$$

*Proof.* See B □

**Corollary 3.2.8.** *We have the following property in* $\widehat{\mathbb{B}}$*:*

$$[n+1, \sum_{i\in k} x_i] \cong \sum_{i\in k} \int^q [q+1, x_i] \times [n, \sum_{j\in k\setminus\{i\}} x_j + q]$$

This leads to a new proof of the following:

**Theorem 3.2.9.** *The chain rule* $(\partial F \circ G)\cdot \partial G \cong \partial(F\circ G)$ *holds in species.*

*Proof.* See B. □

Strangely enough the second property of the above lemma suffices to prove Leibniz rule quite easily. Thus Leibniz rule might actually be rather general.

**Proposition 3.2.10.** *Using property* 2 *above we can also prove Leibniz Rule in species:*

*Proof.*

$$\begin{aligned}
\partial(F\cdot G)(n) &\overset{def}{\cong} \int^{x,y} F_x \times G_y \times [n+1, x+y] \\
&\overset{1}{\cong} \int^{x,y,q} F_x \times G_y \times [q+1, x] \times [n, q+y] + \int^{x,y,q} F_x \times G_y \times [q+1, y] \times [n, x+q] \\
&\overset{2}{\cong} \int^{y,q} F_{q+1} \times G_y \times [n, q+y] + \int^{x,q} F_x \times G_{q+1} \times [n, x+q] \\
&\overset{3}{\cong} \int^{x,y} F_{x+1} \times G_y \times [n, x+y] + \int^{x,y} F_x \times G_{y+1} \times [n, x+y] \\
&\overset{def}{\cong} \partial F \cdot G + F \cdot \partial G
\end{aligned}$$

where we used the following facts at each step:

1. property 2 from above lemma, + and coends commute (both are colimits)
2. density formula in $x, y$ respectively
3. Fubini theorem, renaming of variables

□

### 3.2.3 Structural properties

We give two new characterizations.

**Proposition 3.2.11.** *Consider a* $\circ$*-monoid* $\mathcal{Y}(1) \xrightarrow[e]{} X(k) \xleftarrow[m]{} X \circ X$ *where* $X \in \widehat{\mathbb{B}}$*. The following are equivalent:*

- *an element* $c \in X(k)$
- *a map* $f : X^{\cdot k} \to X$ *such that* 
$$\begin{tikzcd} X^{\cdot k}\circ X \arrow[r, "st_X", "\sim"'] \arrow[d, "f\circ m"'] & (X\circ X)^{\cdot k} \arrow[r, "m^{\cdot k}"] & X^{\cdot k} \arrow[d, "f"] \\ X\circ X \arrow[rr, "m"'] & & X \end{tikzcd}$$
 *commutes*

*Proof.* See B. □

This means such a map $f$ is seen as a $k$-ary operation in the operad. It is indeed just a particular given element in the operad which has $k$ inputs. This allows us to add specific elements in the operad by means of such maps $f$ in the corresponding monoid.

The intuition would then be that a monoid homomorphism which sends such a given map $f$ to a map $f'$ would be an operad homomorphism which sends the associated operator $c$ for $f$ to the operator $c'$ associated to $f'$. This is actually true:

**Theorem 3.2.12.** *Consider two $\circ$-monoids $X$ and $Y$, a $\circ$-monoid homomorphism $h : X \to Y$ and two maps $X^{\cdot k} \xrightarrow{f} X$ and $Y^{\cdot k} \xrightarrow{f'} Y$ satisfying the hypothesis of the previous theorem, second point. Also denote by $c, c'$ the elements in $X(k), Y(k)$ respectively given by $f, f'$ by the previous theorem. The following are equivalent:*

- $h_k(c) = c'$
- *$h$ maps $f$ to $f'$*, i.e. $\begin{array}{ccc} X^{\cdot k} & \xrightarrow{h^{\cdot k}} & X'^{\cdot k} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ X & \xrightarrow[h]{} & X' \end{array}$ *commutes*

*Proof.* See B. □

### 3.2.4 Strengths

As said in the background, we will need strengths to be able to commute the process of substitution and operations. The strength for $\partial$ is in the theory of syntax referred to as avoiding the capture of free variables. We sum up the following known results in the next lemma.

**Lemma 3.2.13.** *In $\widehat{\mathbb{B}}$ we have the following left strengths with $\circ$.*

1. *addition:* $(X + Y) \circ Z \xrightarrow{\cong} (X \circ Z) + (Y \circ Z)$
2. *product:* $(X \cdot Y) \circ Z \xrightarrow{\cong} (X \circ Z) \cdot Y + X \cdot (Y \circ Z)$
3. *differentiation:* $\partial(X) \circ Z \to \partial(X \circ Z)$
4. *multiplication:* $(X \times Y) \circ Z \to (X \circ Z) \times (Y \circ Z)$

*We also have other commutation of operators:*

1. *addition:* $\partial(X + Y) \cdot Z \xrightarrow{\cong} \partial X \cdot Z + \partial Y \cdot Z$
2. *differentiation:* $\partial^2 X \cdot Z \to \partial(\partial X \cdot Z)$
3. *product:* $\partial(X \times Y) \cdot Z \to (\partial X \cdot Z) \times (\partial Y \cdot Z)$

We will use these properties in the next chapter and in the conclusion chapter.

# Chapter 4

# Algebraic structures

## 4.1 Algebraic operations

In the same vein as [16, 5], we would like to see our constructions as initial objects in the category of $F$-monoids for a certain $F$ which carries the desired algebraic operations. In particular we would like the property explained in the Background section of initial algebras as inductive objects.

We work in $\widehat{\mathbb{B}}$ for this section.

$+$ acts as a choice of operation. Indeed, if you consider $\Sigma(-) = A(-)+B(-)$ for two functors $A, B$ representing an operation, then a map $\Sigma X \to X$ is equivalent by the universal property of the coproduct to two maps $AX \to X$ and $BX \to X$. Each of these map thus gives an interpretation of an operator. Another way to see it is that, when constructing iteratively the initial algebra, the coproduct is computed in our case point-wise as the disjoint union $A+B(D) \overset{def}{=} A(D)+B(D)$, for $D$ the set of terms constructed so far. It means the set of terms is the union of terms where you chose either an operation $A$ as the root of a given term or an operation $B$.

$\cdot^k$ gives the arity $k$ of an operator and $\partial^n$ binds $n$ variables. For instance an operator $let$ in a context $\Gamma$, $X \cdot \partial X(\Gamma) \overset{let}{\to} X(\Gamma)$, takes two terms $t_1 \in \Gamma_1, t_2 \in \Gamma_2$, for $\Gamma_1 + \Gamma_2 \cong \Gamma$ and outputs $let(t_1, t_2)$. One can see $\Gamma_1+\Gamma_2 \cong \Gamma$ as the preservation of linearity and it also enables a renaming of variables, in De Bruijn style. Also notice that $t_2$ has a bound variable. Seen differently one could write:

$$\frac{\Gamma_1 \vdash t_1 \qquad \Gamma_2, x \vdash t_2}{\Gamma_1 + \Gamma_2 \vdash \text{let } x = t_1 \text{ in } t_2^*}$$

where $t_2^*$ is $t_2$ with its variables $x_i$ renamed $x_{i+k}$ for $k$ the size of $\Gamma_1$, and where $x$ is the bound variable in $t_2$. The initial algebra will be syntactic, but one could give it an interpretation which respects every established syntactic rule.

Finally we are able to use $\times$ which duplicates data. Indeed $X \times X(n) \overset{def}{=} X(n) \times X(n)$. The context is not divided as with $\cdot$ but rather duplicated. It allows us to deal non linear operations. We are thus able to deal with both linear and non linear operations in their variables.

We will only consider algebraic theories with a finite number of operations. An operation has a finite number of arguments and each argument may bind a finite number of variables. We can sum up these operations in a signature as recalled in the background chapter. In addition to the usual signature, we are able to deal with operations which are not linear thanks to $\times$. To encompass these, we thus consider an endofunctor $\Sigma : \widehat{\mathbb{B}} \to \widehat{\mathbb{B}}$ generated by $Id, \partial, \times, +, \cdot$, where $Id$ is the identity functor on $\widehat{\mathbb{B}}$.

**Lemma 4.1.1.** *$\Sigma$ preserves colimits.*

*Proof.* $Id$ preserves colimits. Colimits commute so $+$ preserves colimits. $\cdot, \times$ have right adjoints so preserve colimits. Colimits are computed point-wise here so $\partial$ preserves them:

$$\begin{aligned} \Big(\partial(colimF_i)\Big)(c) &= (colimF_i)(c+1) \\ &= colim\big(F_i(c+1)\big) \\ &= colim\Big((\partial F_i)(c)\Big) \\ &= (colim\partial F_i)(c) \end{aligned}$$

□

We have the following results given in [16], which extend to our case:

Let $U$ be the forgetful functor from the category of $\Sigma$-algebras to $\widehat{\mathbb{B}}$. This functor has a left adjoint that carries each presheaf $X$ to $TX$, the free $\Sigma$-algebra over $X$, which is computer as $TX = \coprod_{n \in \mathbb{N}} (X + \Sigma)^{(n)}(0)$, where $(-)^{(n)}$ is the iterated composition, $X$ is regarded as a constant endofunctor on $\widehat{\mathbb{B}}$ and $0$ being the presheaf which sends all $n$ to $\varnothing$. Since $X + \Sigma$ preserves colimits, we have $\Sigma TX \cong TX$ and the canonical isomorphism can be taken as the morphism for the algebra.

We define the presheaf $G$ so that for every $\Gamma \in \mathbb{B}$, $G(\Gamma) = \{t | \Gamma \vdash_\Sigma t\}$ where $\vdash_\Sigma$ is given by $\Sigma$, as in the example of a functor associated to a signature and the intuition given in the background chapter, and by the rule $x_1 \vdash_\Sigma x_1$.

**Theorem 4.1.2.** *The presheaf $G$ associated to $\Sigma$ is a free $\Sigma$-algebra on the presheaf of linear variables $\mathcal{Y}(1)$.*

In general, if a $T : \mathcal{C} \to \mathcal{C}$ is a strong monad on a monoidal closed category $\mathcal{C} = (\mathcal{C}, \cdot, I)$, the object $TI$ has a monoid structure. Moreover, we can also show that if $T$ is the free monad on a strong endofunctor $F$ on $\mathcal{C}$, the strength of $F$ extends to a strength of $T$ as a monad.

In our case we then have that $(T\mathcal{Y}(1), \sigma, \eta_{\mathcal{Y}(1)})$ is a monoid in $\widehat{\mathbb{B}}$ where $\eta_{\mathcal{Y}(1)}$ is the universal arrow and $\sigma$ is the unique extension of the unit isomorphism. If we let $\phi_{\mathcal{Y}(1)}$ be a free algebra over $\mathcal{Y}(1)$, we have the following result:

**Theorem 4.1.3.** *$(T\mathcal{Y}(1), \sigma, \eta_{\mathcal{Y}(1)}, \phi_{\mathcal{Y}(1)})$ is an initial $\Sigma$-monoid.*

However we still need to add equations to our theory.

## 4.2 Algebraic equations

We suppose given an endofunctor $\Sigma$ as above. Chose equations that the operations given by $\Sigma$ need to satisfy. Equations are given as commutative diagrams involving operations in $\Sigma$. For instance if $\Sigma X = X + X$ Then one may want that these operations commute. Denote by $a : X \to X, b : X \to X$ these two operations. Then the commutativity of the operations is given by the commutativity of the following diagram

$$\begin{array}{ccc} X & \xrightarrow{a} & X \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[a]{} & X \end{array}$$

We suppose given a finite number of such equations which we call $E$. We consider the category $(\Sigma, E) - Mon$ of $\Sigma$-monoids which satisfy the equations in $E$.

We would like to say the category $(\Sigma, E) - Mon$ has an initial object which can be computed inductively as iterated of $\mathcal{Y}(1) + \Sigma(-)$ as before. It is sufficient that $(\Sigma, E) - Mon$ is cocomplete to be able to do so. Actually this is true by a general theorem from [3]. Our case is close to their example 7.2. It also claims that the forgetful functor $U : (\Sigma, E) - Mon \to \widehat{\mathbb{B}}$ is monadic and the induced monad preserves epimorphisms. We sum up the result in the following theorem:

**Theorem 4.2.1.** *Consider an endofunctor $\Sigma : \widehat{\mathbb{B}} \to \widehat{\mathbb{B}}$ finitely generated by $Id, \partial, \times, +, \cdot$ and a set of equations $E$. We denote by $\Sigma, E) - Mon$ the category of $\Sigma$-monoids which satisfy the equations in $E$. Then $(\Sigma, E) - Mon$ is cocomplete and its initial object is computed inductively as the iterated of $\mathcal{Y}(1) + \Sigma(-)$ over the initial presheaf $0$.*

## 4.3 Examples

The example of the introduction can now be easily treated as a first example.

**Example 4.3.1** (Rooted non-planar binary trees)**.** We can now say formally what was said in the introduction. The operad of rooted non-planar binary trees is the initial object in the category of operads $O$ with algebraic structure given by an operation $O \cdot O \to O$ satisfying the commutativity equation.

A more interesting example is the one of vector spaces. We see how we can characterize the vector space operad and how can the pre-Lie structure can be added as an particular choice of equation.

**Example 4.3.2** (Operads in vector spaces)**.** Given a field $\Bbbk$, take an operad $O$ with operations:

$$Addition : \underline{+} : O \times O \to O$$

$$Scalar\ Multiplication : \underline{\lambda} : O \to O \ (\lambda \in \Bbbk)$$

subject to the usual axioms which can easily be written as commutative diagrams.where $sym$ is the symmetry for $\times$. This gives us a species $O$ in vector spaces. The compatibility of the above two operations with the composition law $\sigma : O \circ O \to O$ says exactly that composition is linear in its first argument. Thus, to get the notion of operad in vector spaces we need to postulate the following compatibility equations:

$$\begin{array}{ccccc} O \circ (O \times O) & \longrightarrow & (O \circ O) \times (O \circ O) & \xrightarrow{\sigma \times \sigma} & O \times O \\ {\scriptstyle O \circ \underline{+}}\downarrow & & & & \downarrow{\scriptstyle \underline{+}} \\ O \circ O & & \xrightarrow[\sigma]{\qquad\qquad} & & O \end{array} \qquad \begin{array}{ccc} O \circ O & \xrightarrow{\sigma} & O \\ {\scriptstyle O \circ \underline{\lambda}}\downarrow & & \downarrow{\scriptstyle \underline{\lambda}} \\ O \circ O & \xrightarrow[\sigma]{} & O \end{array}$$

Further algebraic structure can be added on top of that. For instance pre-Lie structure is added by means of an operation $\underline{*} : O \cdot O \to O$ which is bilinear:

$$
\begin{tikzcd}
(O \times O) \cdot O \arrow[r] \arrow[d, "\underline{+} \cdot O"'] & (O \cdot O) \times (O \cdot O) \arrow[r, "\underline{*} \times \underline{*}"] & O \times O \arrow[d, "\underline{+}"] \\
O \cdot O \arrow[rr, "\underline{*}"'] & & O
\end{tikzcd}
\qquad
\begin{tikzcd}
O \cdot O \arrow[r, "\underline{*}"] \arrow[d, "\underline{\lambda} \cdot O"'] & O \arrow[d, "\underline{\lambda}"] \\
O \cdot O \arrow[r, "\underline{*}"'] & O
\end{tikzcd}
$$

$$
\begin{tikzcd}
O \cdot (O \times O) \arrow[r] \arrow[d, "O \cdot \underline{+}"'] & (O \cdot O) \times (O \cdot O) \arrow[r, "\underline{*} \times \underline{*}"] & O \times O \arrow[d, "\underline{+}"] \\
O \cdot O \arrow[rr, "\underline{*}"'] & & O
\end{tikzcd}
\qquad
\begin{tikzcd}
O \cdot O \arrow[r, "\underline{*}"] \arrow[d, "O \cdot \underline{\lambda}"'] & O \arrow[d, "\underline{\lambda}"] \\
O \cdot O \arrow[r, "\underline{*}"'] & O
\end{tikzcd}
$$

and satisfies the pre-Lie axiom:

$$
\begin{tikzcd}
(O \cdot O) \cdot O \arrow[r, "h"] \arrow[d, "k"'] & O \times O \arrow[d, "\underline{+}"] \\
O \times O \arrow[r, "\underline{+}"'] & O
\end{tikzcd}
$$

where $h, k$ are respectively given as the unique map given by the universal property of the Cartesian product in the following diagrams, where $sym : O \cdot O \xrightarrow{\cong} O \cdot O$ denotes the natural isomorphism for the symmetry:

$$
\begin{tikzcd}
(O \cdot O) \cdot O \arrow[r, "sym \circ \alpha", "\sim"'] \arrow[d, "\underline{\star} \cdot O"'] \arrow[dddrrr, dashed, "h"] & O \cdot (O \cdot O) \arrow[r, "O \cdot \underline{\star}"] & O \cdot O \arrow[r, "\underline{\star}"] & O \\
O \cdot O \arrow[d, "\underline{\star}"'] & & & \\
O & & & \\
& & & O \times O \arrow[uuu, "\pi_2"'] \arrow[lll, "\pi_1"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(O \cdot O) \cdot O \arrow[r, "\alpha"] \arrow[d, "\alpha \circ sym \circ \alpha"'] \arrow[ddddrrr, dashed, "k"] & O \cdot (O \cdot O) \arrow[r, "O \cdot \underline{\star}"] & O \cdot O \arrow[r, "\underline{\star}"] & O \\
(O \cdot O) \cdot O \arrow[d, "\underline{\star} \cdot O"'] & & & \\
O \cdot O \arrow[d, "\underline{\star}"'] & & & \\
O & & & \\
& & & O \times O \arrow[uuuu, "\pi_2"'] \arrow[lll, "\pi_1"]
\end{tikzcd}
$$

It can be seen in more concrete terms as $(x \star y) \star z + x \star (z \star y) = (x \star z) \star y + x \star (y \star z)$.

The pre-Lie operad is the initial object in the category of operads with the above algebraic structure.

We are allowed to use the $\partial$ operation to define our algebraic theories. This gives rise to new kinds of operads. For example one can consider the operad of linear lambda terms as follows.

**Example 4.3.3** (Linear lambda calculus)**.** The operad of linear lambda terms $L$ is syntactically specified by the following inductive rules

$$
\frac{}{x_1 \in L(1)} \qquad \frac{t \in L(n+1)}{\lambda x_{n+1}.t \in L(n)} \qquad \frac{t_1 \in L(n) \qquad t_2 \in L(m)}{u(v^{(n)}) \in L(n+m)}
$$

where $x_i$ are variables, $\lambda$ a binding operator and $v^{(n)}$ the result of renaming every free variable $x_i$ by $x_{n+i}$ and a proper renaming of bounded variable to avoid the capture of free variables. The composition law maps $(t, v_1, \ldots, v_n) \in L(n) \times L(m_1) \times \cdots \times L(m_n)$ to the result in $L(m_1 + \cdots + m_n)$ of substituting the free variable $x_i$ in $t$ with $v_i^{(m_1+\cdots+m_{i-1})}$ without capturing free variables.

The operad $L$ is the initial object in the category of operads $O$ with algebraic structure consisting of the operations:

$$\text{Application } O \cdot O \to O$$

$$\text{Abstraction } \partial(O) \to O$$

# Chapter 5

# Conclusion

## 5.1 Review

Using a particular closed monoidal category of presheaf, the category of species, we characterized operads as $\circ$-monoids and gave additional constructions on the category of species: the canonicity of the isomorphism for Leibniz rule, an abstract proof for the chain rule, and two characterisations related to the $\circ$-monoid structure. Then we showed how to add equations and operations to $Set$ operads to recover usual kinds of operads people use. In particular we used universal algebra which allow to go to the theory of models, and we are able to add the binding operation to the operadic world.

We only treated the simultaneous substitution case. Similar things could be done with single substitution. In addition one could add a notion of metavariable to encompass the notion of substitution at internal nodes of operads. We discuss this possible future work below.

## 5.2 Single substitution

There is an equivalent representation of operads due to May in terms of $\circ_i : O(p) \times O(q) \to O(p+q-1)$ which sends two terms $t, t'$ to the term where we plugged $t'$ at the input $i$ of $t$.

One may start dealing with the single substitution case as follows. We need an operator to model this single substitution. The idea is somehow to choose an input where to plug two terms. For example it may be chosen to be the last one. We can then think of the context extension $\partial$. As we work with symmetric operads this will end up to be equivalent to the representation above with $\circ_i$. However we won't end up with a monoidal category and a tensor product, but instead a lax version of it, called skew-monoidal.

**Definition 5.2.1** (Skew monoidal). *A category $(\mathcal{C}, \star, I, \alpha, \lambda, \rho)$ where $\star : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $I \in \mathcal{C}$ and $\alpha, \beta, \lambda$ are three natural transformations, is said to be left skew-monoidal when we have the following diagrams commute:*

$$\begin{tikzcd}
I \arrow[rr, "1_I"] \arrow[dr, "\rho_I"'] & & I \\
& I \star I \arrow[ur, "\lambda_I"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \star (A \star B) \arrow[rr, "\alpha_{I,A,B}"] & & (I \star A) \star B \\
& I \arrow[ul, "\lambda_{A \star B}"] \arrow[ur, "\lambda_A \star B"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=small]
A \star (B \star I) \arrow[rr, "\alpha_{A,B,I}"] \arrow[dr, "A\star\rho_B"'] & & (A \star B) \star I \arrow[dl, "\rho_{A\star B}"] \\
& I &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \star (I \star B) \arrow[r, "\alpha_{A,I,B}"] & (A \star I) \star B \arrow[d, "\rho_A \star B"] \\
A \star B \arrow[u, "A\star\lambda_B"] \arrow[r, equal] & A \star B
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& A \star (B \star (C \star D)) \arrow[dl, "A\star\alpha_{B,C,D}"'] \arrow[dr, "\alpha_{A,B,C\star D}"] & \\
A \star ((B \star C) \star D)) \arrow[d, "\alpha_{A,B\star C,D}"'] & & (A \star B) \star (C \star D) \arrow[d, "\alpha_{A\star B,C,D}"] \\
(A \star (B \star C)) \star D \arrow[rr, "\alpha_{A,B,C\star D}"'] & & ((A \star B) \star C) \star D
\end{tikzcd}
$$

**Theorem 5.2.2.** *We define for* $A, B \in \widehat{\mathbb{B}}$*:*

$$A \star B \overset{def}{=} \partial(A) \cdot B$$

*Then* $(\widehat{\mathbb{B}}, \star, \mathcal{Y}(1))$ *is skew-monoidal.*

We may now define $\star$-monoids in the same way we considered $\circ$-monoids before. The intuition is the following: the extra variable given by $\partial$ is the one where we substitute. Things could be redone using universal algebra but is reached here. Note that we will need strengths between $\partial$ and $+, \cdot, \times$ which we have.

In more concrete terms we have in this structure the following. For a binary operation $f$ and two unary operations $g, h$ in this setting, in general $f(I \otimes g)(h \otimes I)$ won't be equal to $f(h \otimes I)(I \otimes g)$. It means grafting $g$ to $f$ and then $h$ is not the same as grafting $h$ and then $g$. When such a possibility is given we are close to the notion of premonoidal category, as defined by Power and Robinson in [15]. It is related to side effects for the following reason. Side effects can be represented by operations, as $f, g, h$ above, which does not necessary commute. The order in which effects occur matters and this is why our skew-structure might be interesting to study further.

## 5.3 Internal substitution

There is another operation in operads which consists in the substitution of an internal node. We can give an interpretation of our terms but cannot perform such an operation. However the work of Fiore [2] has shown a way to add, in $\widehat{\mathcal{F}}$, a second internal notion of substitution, given by the Kleisli composition, which allows to add new kinds of variables at internal nodes, called meta-variables, and substitute them by a term which may also contain meta-variables.

This still won't be directly possible in $\widehat{\mathbb{B}}$. Indeed there is no strength from $\circ$ over $\times$, which is needed for the meta-variable construction in $\widehat{\mathcal{F}}$. New work has shown a way to overcome this problem. Also note that meta-variables can be used several times in a term, which we don't have in operads. In operads the equivalent of meta-variables are affine ($\leqslant 1$ of each meta-variable per term).

We also found a way to overcome this problem by getting one level higher in abstraction: the work is done in $\widehat{\mathbb{B}}^{!\mathbb{B}}$ instead of $\widehat{\mathbb{B}}$, where ! sends a category to its the free symmetric monoidal category. For instance we had $!1 = \mathbb{B}$. However, there still remains work to be done to check every detail and to add equations on top of that. Going to $\widehat{\mathbb{B}}^{!\mathbb{B}}$ is still going to be interesting because it is a particular generalised species, which were introduced and developed by Fiore and al. in [6], and it thus provides an example for his theory and a general context to our work. In addition, doing so allows us to choose whether we want linearity in meta-variables or not, the former being operadic and the latter being something new.

# Appendix A

# Categorical background

**Definition A.0.1** (Coend)**.** *The coend of a functor $S : \mathcal{C}^{op} \times \mathcal{C} \to X$ is a pair $(d, \zeta : S \to d)$ where $d$ is an object of $X$ and $\zeta$ is a dinatural transformation (a wedge), universal among dinatural transformations from S to a constant. The object $d$ (when it exists, unique up to isomorphism) will be written with an integral sign as follows:*

$$S(c,c) \xrightarrow{\zeta_c} \int^c S(c,c) = d$$

We will be working with coends living in $Set$, which can be seen as some quotient over a coproduct. We explain the intuition for this below. For every other wedge $(A, \alpha)$ and map $f : c \to d$ one has the following diagram:

$$\begin{tikzcd}
 & S(c,c) \arrow[dr, "\zeta_c"] \arrow[drr, bend left, "\alpha_c"] & & \\
S(c,d) \arrow[ur, "{S(c,f)}"] \arrow[dr, "{S(f,d)}"'] & & \int^e S(e,e) \arrow[r, dashed, "\exists! \beta_{c,d}"] & A \\
 & S(d,d) \arrow[ur, "\zeta_d"'] \arrow[urr, bend right, "\alpha_d"'] & &
\end{tikzcd}$$

This means whenever there is a map $f : c \to d$ you can link S(c,c) and S(d,d) as in the square above. The commutativity of the diagram tells you somehow that $S(c,c)$ and $S(d,d)$ must behave the same when sent to the coend, so that you must identify them. Calculations can confirm this intuition and that explains why we work with class of equivalences. In addition it explains when two elements must be identified, *i.e.* when two elements represent the same class of equivalence.

There are theorems for (co)ends with parameters and a theorem which allows to commute integral symbols called Fubini theorem for obvious reasons. See [12] for more details.

There is a well-known and very useful lemma :

**Lemma A.0.2.** *Adjoints $F \dashv G$ preserve limits and colimits.*

A (co)nd is a (co)limit — constructed as a (co)limit over the so-called *subdivision category*, a very syntactic construction to make the universal properties match up.

However, this does give us the following, because hom-functors preserve (and reverse) colimits:

$$\Big[x, \int_c S(c,c)\Big] \cong \int_c [x, S(c,c)]$$

$$\Big[\int^c S(c,c), x\Big] \cong \int_c [S(c,c), x]$$

Note that the last integral is an end.

**Lemma A.0.3** (Coends commute with products in a Cartesian closed category)**.** *Let $\mathcal{C}$ be a Cartesian closed category and $S : \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{C}$ a functor. Then coends pass through products:*

$$\int^c \big(S(c,c) \times P\big) \cong \Big(\int^c S(c,c)\Big) \times P$$

*Proof.* The functor $- \times P$ has a right adjoint in a closed Cartesian category, so preserves colimits, and in particular coends. □

**Definition A.0.4.** *The coherence laws for $\cdot$ are given by the following:*

1. $\alpha_{A,B,C} : A \cdot (B \cdot C) \to (A \cdot B) \cdot C$ *is given for all $d \in \mathcal{C}$ by:*

$$\big[x, y, p, [z, t, q, r, f], g\big] \mapsto \big[x + z, t, [x, z, p, q, id], r, (id_x \otimes f)g\big]$$

2. $\lambda_A : A \to \mathcal{Y}(I) \cdot A$ *is given for all $d \in \mathcal{C}$ by:*

$$A(d) \ni p \mapsto [d, I, p, id_I, id_d] \in \int^{x,y} A_x \times [y, I] \times [d, x \otimes y]$$

*as* $d \otimes I \cong d$

3. $\rho_A : A \cdot \mathcal{Y}(I) \to A$ *is given for all $d \in \mathcal{C}$ by:*

$$\int^{x,y} A_x \times [y, I] \times [d, x \otimes y] \ni [x, y, p, f, g] \mapsto A_g(A_{x \otimes f}(p)) \in A(d)$$

*as* $x \otimes I \cong x$

**Definition A.0.5.** *The coherence laws for $\circ$ are given by the following:*

1. $\alpha_{A,B,C} : A \circ (B \circ C) \to (A \circ B) \circ C$ *is given for all $d \in \mathcal{C}$ by:*

$$\Big[n, p, \Big[m_1 \dots m_n, \big[k_1, p_1, [x^1_1 \dots x^1_{k_1}, p^1_1 \dots p^1_{k_1}, f^1]\big], \dots, \big[k_n, p_n, [x^n_1 \dots x^n_{k_n}, p^n_1 \dots p^n_{k_n}, f^n]\big], g\Big]\Big]$$

$$\mapsto \Big[\sum_{i \in n} k_i, \big[n, p, [k_1, \dots, k_n, id]\big], \big[x^1_1 \dots x^n_{k_n}, p^1_1 \dots p^n_{k_n}, (\oplus_{i \in n} f^i)g\big]\Big] \in (A \circ B) \circ C(d)$$

2. $\lambda_A : A \to \mathcal{Y}(1) \circ A$ *is given for all $d \in \mathcal{C}$ by:*

$$A(d) \ni p \mapsto \big[1, id_1, [d, p, id_d]\big] \in \int^k [k, 1] \times A^{\cdot k}(d)$$

3. $\rho_A : A \circ \mathcal{Y}(1) \to A$ *is given for all $d \in \mathcal{C}$ by:*

$$\int^k A_k \times \mathcal{Y}(1)^{\cdot k}(d) \ni \big[k, p, [1 \dots 1, id_1, \dots, id_1, f]\big] \mapsto A_f(p) \in A(d)$$

# Appendix B

# Proofs

*Proof of theorem 3.2.2.* Take a $\circ$-monoid $(X, e, m)$. We define an operad $(\{Y_n\}_{n\in\mathbb{N}}, I, \gamma, \cdot)$ as follows:

- For all $n \in \mathbb{N}, Y_n \overset{def}{=} X(n)$
- $I \overset{def}{=} e_1(id_1)$
- $\gamma(p; p_1 \dots p_k) \overset{def}{=} m_j\Big(\big[k, p, [j_1 \dots j_k, p_1 \dots p_k, id]\big]\Big)$ where $j = \sum_{i\in k} j_i$
- $\sigma \in \mathcal{S}_j$ acts on $Y_j$ by $p \mapsto X_\sigma(p)$.

$$
\begin{aligned}
\gamma\big(\gamma(p; d_1 \dots d_k); e_1 \dots e_j\big) &\overset{def}{=} m_n\Big(\Big[j, m_j\big(\big[k, p, [j_1 \dots j_k, d_1 \dots d_k, id]\big]\big), [x_1 \dots x_j, e_1 \dots e_j, id]\Big]\Big) \\
&= m_n (X \circ m)_n \alpha_n^{-1}\Big(\Big[j, \big[k, p, [j_1 \dots j_k, d_1 \dots d_k, id]\big], [x_1 \dots x_j, e_1 \dots e_j, id]\Big]\Big) \\
&= m_n (X \circ m)_n \Big(\Big[k, p,\ \big[j_1, d_1, [x_1 \dots x_{j_1}, e_1 \dots e_{j_1}, id]\big], \\
&\qquad \vdots \\
&\qquad , \big[j_k, d_k, [x_{j_1+\dots+j_{k-1}+1} \dots x_{j_k}, e_{j_1+\dots+j_{k-1}+1} \dots e_{j_k}, id]\big]\Big]\Big) \\
&= m_n\Big(\Big[k, p,\ m_{n_1}\big([j_1, d_1, [x_1 \dots x_{j_1}, e_1 \dots e_{j_1}, id]]\big), \\
&\qquad \vdots \\
&\qquad , m_{n_k}\big(\big[j_k, d_k, [x_{j_1+\dots+j_{k-1}+1} \dots x_{j_k}, e_{j_1+\dots+j_{k-1}+1} \dots e_{j_k}, id]\big]\big)\Big]\Big) \\
&\overset{def}{=} \gamma(p; f_1 \dots f_k)
\end{aligned}
$$

where we used:

1. associativity of $m$
2. definition of $\alpha$
3. definition of $\circ$

$$
\begin{aligned}
\gamma(I,p) &\overset{def}{=} m_j\Big([1, e_1(id_1), [j,p,id]]\Big)\\
&= m_j\Big((e\circ X)_j([1, id_1, [j,p,id]])\Big)\\
&= \lambda_j^{-1}\Big([1, id_1, [j,p,id]]\Big)\\
&\overset{def\ \lambda}{=} p
\end{aligned}
$$

where we used the coherence law of $m$ with $\lambda$ and the definition of $\circ$.

$$
\begin{aligned}
\gamma(p, I^k) &\overset{def}{=} m_k\Big([k, p, [1\ldots 1, e_1(id_1)\ldots e_1(id_1), id_k]]\Big)\\
&= m_k\Big((X\circ e)_k([k,p,[1\ldots 1, id_1\ldots id_1, id_k]])\Big)\\
&= \rho_k\Big([k,p,[1\ldots 1, id_1\ldots id_1, id_k]]\Big)\\
&= p
\end{aligned}
$$

where we used the definition of $\circ$, the coherence law of $m$ with $\rho$ and the definition of $\rho$.

$$
\begin{aligned}
\gamma(p; d_1\cdot\tau_1\ldots d_k\cdot\tau_k) &\overset{def}{=} m_j\Big([k,p,[j_1\ldots j_k, X_{\tau_1}(d_1)\ldots X_{\tau_k}(d_k), id]]\Big)\\
&= m_j\Big([k,p,[j_1\ldots j_k, d_1\ldots d_k, \otimes_{i\in k}\tau_i]]\Big)\\
&= X_{\otimes_{i\in k}\tau_i}\Big(m_j([k,p,[j_1\ldots j_k, d_1\ldots d_k, id]])\Big)\\
&\overset{def}{=} \gamma(p_1; d_1\ldots d_k)\cdot(\otimes_{i\in k}\tau_i)
\end{aligned}
$$

where we used the definition of the class of equivalence in coends and the definition of the right action on $Y$.

$$
\begin{aligned}
\gamma(p\cdot\sigma; d_1\ldots d_k) &\overset{def}{=} m_j\Big([k, X_\sigma(p), [j_1\ldots j_k, d_1\ldots d_k, id]]\Big)\\
&= m_j\Big([k,p,[j_{\sigma^{-1}(1)}\ldots j_{\sigma^{-1}(k)}, d_{\sigma^{-1}(1)}\ldots d_{\sigma^{-1}(k)}, \widehat{\sigma}(id)]]\Big)\\
&= \gamma(p; d_{\sigma^{-1}(1)}\ldots d_{\sigma^{-1}(k)})\cdot\widehat{\sigma}(j_1\ldots j_k)
\end{aligned}
$$

where we used the definition of the class of equivalence in coends, the definition of $\gamma$ and of the right action on $Y$.

Take an Operad $(\{Y_n\}_{n\in\mathbb{N}}, I, \gamma, \cdot)$. We define a $\circ$-monoid $(X, e, m)$ as follows:

- $X \in \widehat{\mathbb{B}}$ is given by the previous lemma
- $e$ is given by the Yoneda lemma on $I$
- for all $d\in\mathbb{N}, m_d \overset{def}{=} \begin{cases} \int^m X_m \times X^{\widehat{\otimes} m}(d) & \to & X(d)\\ [m,p,[x_1\ldots x_m, p_1\ldots p_m, f]] & \mapsto & \gamma(p; p_1\ldots p_m)\cdot f\end{cases}$

By axioms 3.1 and 3.2 of operads we have that $m_d$ is well defined for all $d$. For $g : d \xrightarrow{\cong} d'$ we have $X_g(\gamma(p;p_1\ldots p_m)\cdot f) \overset{def}{=} (\gamma(p;p_1\ldots p_m)\cdot f)\cdot g = \gamma(p;p_1\ldots p_m)\cdot(fg)$ by the definition of the right action, so $m$ is natural.

-$\lambda m(e\circ X)) \stackrel{?}{=} id$

$$
\begin{aligned}
&\big[1, id_1, [x,p,f]\big]\\
&\stackrel{e\circ X}{\mapsto} \big[1, I, [x,p,f]\big]\\
&\stackrel{m}{\mapsto} \gamma(I,p)f = p\cdot f = X_f(p)\\
&\stackrel{\lambda}{\mapsto} \big[1, id_1, [x, X_f(p), id]\big]\\
&= \big[i, id_1, [x,p,f]\big]
\end{aligned}
$$

by definition of the class of equivalence in the coend.

-$\rho^{-1}m(X\circ e) \stackrel{?}{=} id$

$$
\begin{aligned}
&\big[n,p,[1\dots 1, id_1\dots id_1, f]\big]\\
&\stackrel{X\circ e}{\mapsto} \big[n,p,[1\dots 1, I\dots I, f]\big]\\
&\stackrel{m}{\mapsto} \gamma(p, I^n)f\\
&\stackrel{ax.2.2}{=} p\cdot f \stackrel{def}{=} X_f(p)\\
&\stackrel{\rho^{-1}}{\mapsto} \big[n,p,[1\dots 1, id_1\dots id_1, f]\big]
\end{aligned}
$$

-$m(m\circ X)\alpha \stackrel{?}{=} (X\circ m)$

$$
\begin{aligned}
&\Big[n,p,\Big[x_1\dots x_n, \big[n_1,p_1,[y_1^1\dots y_{n_1}^1, p_1^1\dots p_{n_1}^1, f^1]\big],\dots,\big[n_n,p_n,[y_1^n\dots y_{n_n}^n, p_1^n\dots p_{n_n}^n, f^n]\big], g\Big]\Big]\\
&\xmapsto{\alpha} \Big[\sum_{i\in n} n_i, \big[n,p,[n_1\dots n_n, p_1\dots p_n, id]\big], \big[y_1^1\dots y_{n_n}^n, p_1^1\dots p_{n_n}^n, (\otimes_{i\in n} f^i)g\big]\Big]\\
&\xmapsto{m\circ X} \Big[\sum_{i\in n} n_i, \gamma(p;p_1\dots p_m), \big[y_1^1\dots y_{n_n}^n, p_1^1\dots p_{n_n}^n, (\otimes_{i\in n} f^i)g\big]\Big]\\
&\xmapsto{m} \gamma\big(\gamma(p;p_1\dots p_n); p_1^1\dots p_{n_n}^m\big)\cdot\big((\otimes_{i\in n} f^i)g\big)
\end{aligned}
$$

$$
\begin{aligned}
&\Big[n,p,\Big[x_1\dots x_n,\ \big[n_1,p_1,[y_1^1\dots y_{n_1}^1, p_1^1\dots p_{n_1}^1, f^1]\big],\\
&\qquad\qquad\vdots\\
&\qquad\qquad \big[n_n,p_n,[y_1^n\dots y_{n_n}^n, p_1^n\dots p_{n_n}^n, f^n]\big]\Big]\Big]\\
&\xmapsto{X\circ m} \Big[n,p,\Big[x_1\dots x_n,\ \gamma(p_1;p_1^1\dots p_{n_1}^1)\cdot f^1,\\
&\qquad\qquad\vdots\\
&\qquad\qquad \gamma(p_n;p_1^n\dots p_{n_n}^n)\cdot f^n, g\Big]\Big]\\
&\xmapsto{m} \gamma\big(p;\gamma(p_1;p_1^1\dots p_{n_1}^1)\cdot f^1,\dots,\gamma(p_n;p_1^n\dots p_{n_n}^n)\cdot f^n\big)\cdot g\\
&\stackrel{axiom\ 3.2}{=} \Big(\gamma\big(p;\gamma(p_1;p_1^1\dots p_{n_1}^1),\dots,\gamma(p_n;p_1^n\dots p_{n_n}^n)\big)\cdot g\Big)\cdot\big(\otimes_{i\in n} f^i\big)
\end{aligned}
$$

And it's finally fine by axiom 3.1 and the fact that $\cdot$ is a right action.

Take a monoid homomorphism $h : (X, e, m) \to (Y, e', m')$. Then $x$ is an operad homomorphism for the associated operads of $X$ and $Y$. Indeed $h_1(e_1(id_1)) = e'_1(id_1)$ by the coherence law of $h$ with the unit of the monoid. Also

$$\begin{aligned} h\big(\gamma(p; p_1, \dots, p_n)\big) &\overset{def}{=} h\Big(m\big([n, p, [x_1 \dots x_n, p_1 \dots p_n, id]]\big)\Big) \\ &= m'\big([n, h(p), [x_1 \dots x_n, h(p_1) \dots h(p_n), id]]\big) \\ &\overset{def}{=} \gamma'\big(h(p); h(p_1) \dots h(p_n)\big) \end{aligned}$$

by the the coherence law of $h$ with the multiplication of the monoid. Finally $h(p \cdot \sigma) \overset{def}{=} h(X_\sigma(p)) = Y_\sigma(h(p)) \overset{def}{=} h(p) \cdot' \sigma$ because $h$ is a natural transformation.

The process is very similar for the other way and is functorial. One can easily show both process are inverse to each other on objects and morphisms. This shows we have an isomorphism of categories. □

*Proof of theorem 3.2.2.* We are going to look at more precisely what are the steps in the proof of the previous lemma doing.

Let's first look at this:

$$\frac{Z \xrightarrow{\varphi} U(- + c)}{Z \cdot \mathcal{Y}(c) \xrightarrow{\psi} U}$$

Given $\varphi$ we define $\tilde{\varphi} : Z \cdot \mathcal{Y}(c) \to U$ such that for every $d$:

$$\tilde{\varphi}_d \colon \begin{cases} \int^x Z_x \times [d, x \otimes c] & \to & U(d) \\ [x, a, f] & \mapsto & U_f(\varphi_x(a)) \end{cases}$$

Given $\psi$ we define $\hat{\psi} : Z \to U(- \otimes c)$ such that for every $d$:

$$\widehat{\psi_d} : \begin{cases} Z(d) & \to & U(d \otimes c) \\ p & \mapsto & \psi_{d \otimes c}([d, p, id]) \end{cases}$$

For all $g : x \to y, p \in U(y + c)$ we have $[x, U_{g+c}(p), f] = [y, p, (g + id_c)f]$ by definition and the following diagram commutes because $\phi$ is natural and $U$ a functor:

$$\begin{tikzcd} Z(y) \arrow[r, "\phi_y"] \arrow[d, "Z_g"'] & U(y + c) \arrow[d, "Ug+c"] \arrow[dr, "U_{(g+c)f}"] & \\ Z(x) \arrow[r, "\phi_x"'] & U(x + c) \arrow[r, "U_f"'] & U(d) \end{tikzcd}$$

So $U_{(g+c)f}(\phi_y(p)) = U_f(\phi_x(U_{g+c}(p)))$ and $\tilde{\varphi}$ is well defined.

$U_f U_g = U_{gf}$ so $\tilde{\varphi}$ is natural.

$\psi$ is natural so for any $f : d' \to d$ we have $U_{f \otimes c}(\psi_{d \otimes c}([d, p, id])) = \psi_{d' \otimes c}([d, p, f]) = \psi_{d' \otimes c}([d'_f(p), id])$. This means $\hat{\psi}$ is natural.

$\widehat{\widetilde{\psi}} : [x, a, f] \mapsto U_f(\widehat{\psi_x}(a)) = U_f(\psi_{x\otimes c}([x, a, id])) = \psi_d([x, a, f])$ because $\psi$ is natural.

$\widehat{\widetilde{\varphi}} : p \mapsto \widetilde{\varphi_{d\otimes c}}([d, p, id]) = U_{id}(\varphi_d(p)) = \varphi_d(p)$

So we found exactly what is the adjunction doing.

We also have the evaluation map from the adjunction $\varepsilon : (\mathcal{Y}(c) \multimap X) \cdot \mathcal{Y}(c) \to X$ or equivalently by the density formula, for all $d \in \mathcal{B}$:

$$\varepsilon_d : \begin{cases} \displaystyle\int^{x,y} X(x \otimes c) \times [x, c] \times [d, x \otimes cy] & \to & X(d) \\ \qquad\qquad [x, y, p, f, g] & \mapsto & X_g(X_{id_a \otimes f}(p)) \end{cases}$$

We now calculate what the map becomes throughout all the steps given by the proof that there exists a canonical map for the Leibniz rule. We only give the calculation for the right part $\big((\mathcal{Y}(c) \multimap X) \cdot \mathcal{Y}(c)\big) \cdot Y \to X \cdot Y$ of the canonical map.

$[x, y, [a, b, p, f, g], q, h] \overset{\varepsilon_X \cdot id}{\mapsto} [x, y, X_g(X_{id_a \otimes f}(p)), q, h]$ by definition (of the day tensor product acting on morphism and of the evaluation map)

Then the class of equivalence on the left becomes:

1. $[a, b + y, p, [b, y, f, q, id_{b+y}], (g \otimes id_y)h]$
2. $[a, b + y, p, [y, b, q, f, \tau : b + y \to y + b], (g \otimes id_y)h]$ where $\tau$ is the block transposition
3. $[a + y, b, [a, y, p, q, id_{a+y}], f, (id_a \otimes \tau)(g \otimes id_y)h]$
4. $[a + y, [a, y, p, q, id_{a+y}], (id_{a+y} \otimes f)(id_a \otimes \tau)(g \otimes id_y)h]$ by density formula
5. $[a + y, [a, y, p, q, id_{a+y}], (id_a \otimes \tau')(id_a \otimes f \otimes id_y)(g \otimes id_y)h]$ where $\tau' : c + y \to y + c$ is the block transposition

Notice that by definition of the class of equivalence we have the following equality on the right:

$$\begin{aligned} [x, y, X_g(X_{id_a \otimes f}(p)), q, h] &= [a + b, y, (X_{id_a \otimes f}(p), q, (g \otimes id_y)h] \\ &= [a + c, y, p, q, (id_a \otimes f \otimes id_y)(g \otimes id_y)h] \end{aligned}$$

As this is true for every $h$ and we only consider bijections, we thus get a map:

$$[a + y, [a, y, p, q, id_{a+y}], id_{a+y}] \overset{\psi_d}{\mapsto} [a + c, y, p, q, id_a \otimes \tau'^{-1}]$$

Now using the calculation of the adjunction above, we get the map we defined to prove Leibniz rule is true in species. It thus gives an isomorphism. □

*Proof of lemma 3.2.7.*

1. A map $f : m + 1 \to h$ amounts to an element $j = f(1))$ and a map $\hat{f} : m \to h \backslash \{j\}$. This process is natural in $m$.
2. We only sketch the proof. We define two natural transformations and show they are inverse to each other. We define $\alpha : [- + 1, a + b] \to \int^q [q + 1, a] \times [-, q + b] + \int^q [q + 1, b] \times [-, a + q]$ so that for all $n$:

   $$\alpha_n(f : n + 1 \to a + b) = [a \backslash \{x\}, \sigma, \hat{f}]$$

   where

- $x \overset{def}{=} f(n+1) \in a$
- $\star$ the only element in $1$
- $a \backslash \{x\} \overset{def}{=} a-1$ where we rename everything depending on this $a-1$ properly
- $\sigma : a\backslash\{x\} + 1 \to a$ sends $i$ to $i$ and $\star$ to $x$
- $\hat{f}$ sends $i$ to $\sigma^{-1}f(i)$ if $f(i) \in a$ and $f(i)$ otherwise

One can do a similar definition when $f(n+1) \in b$.

Then by the universal property of the coproduct, it is equivalent to give a map $A + B \to C$ and to give two maps $A \to C$, $B \to C$, so we only give the first part of the map which will be the reciprocal of $\alpha$, defined for all $n \in \mathbb{B}$ as follows:

$$\beta_n : \begin{array}{rcl} \displaystyle\int^q [q+1, a] \times [n, q+b] & \to & [n+1, a+b] \\ [q, f, g] & \mapsto & [(f \otimes id_b)\tau g^*] \end{array}$$

where

- $g^* : n+1 \to q+b+1$ sends $i \in q+b$ to $g(i) \in q+b$ and $\star$ to $\star \in 1$
- $\tau : b+1 \to 1+b$ sends $i \in b$ to $i \in b$ and $\star$ to $\star \in 1$

One can show these maps are well defined, natural and inverse to each other.

$\square$

*Proof of theorem 3.2.9.*

$$
\begin{aligned}
(\partial F \circ G)\cdot \partial G &\overset{def}{\cong} \int^{p,q}\int^{m} F_{m+1} \times \int^{x\in\widehat{\mathbb{B}}^m} \prod_{i\in m} G_{x_i} \times [p, \sum_{j\in m} x_j] \times G_{q+1} \times [n, p+q] \\
&\overset{①}{\cong} \int^{q}\int^{m} F_{m+1} \times \int^{x\in\widehat{\mathbb{B}}^m} \prod_{i\in m} G_{x_i} \times G_{q+1} \times [n, \sum_{i\in m} x_i + q] \\
&\overset{②}{\cong} \int^{m} F_{m+1} \times \int^{q,x\in\widehat{\mathbb{B}}^m} \prod_{i\in m} G_{x_i} \times G_{q+1} \times [n, \sum_{i\in m} x_i + q] \\
&\overset{③}{\cong} \int^{m}\int^{h} F_{h} \times [m+1,h] \times \int^{p,x\in\widehat{\mathbb{B}}^m} \prod_{i\in m} G_{x_i} \times G_p \times \int^{q} [q+1,p] \times [n, \sum_{i\in m} x_i + q] \\
&\overset{④}{\cong} \int^{h} F_{h} \times \int^{m} [m+1,h] \times \int^{p,x\in\widehat{\mathbb{B}}^m} \prod_{i\in m} G_{x_i} \times G_p \times \int^{q} [q+1,p] \times [n, \sum_{i\in m} x_i + q] \\
&\overset{⑤}{\cong} \int^{h} F_{h} \times \int^{m} \sum_{j\in h} [m, h\backslash\{j\}] \times \int^{p,x\in\widehat{\mathbb{B}}^m} \prod_{i\in m} G_{x_i} \times G_p \times \int^{q} [q+1,p] \times [n, \sum_{i\in m} x_i + q] \\
&\overset{⑥}{\cong} \int^{h} F_{h} \times \sum_{j\in h} \int^{p,x\in\widehat{\mathbb{B}}^{h\backslash\{j\}}} \prod_{i\in h\backslash\{j\}} G_{x_i} \times G_p \times \int^{q} [q+1,p] \times [n, \sum_{i\in h\backslash\{j\}} x_i + q] \\
&\overset{⑦}{\cong} \int^{h} F_{h} \times \sum_{j\in h} \int^{x\in\widehat{\mathbb{B}}^{h}} \prod_{i\in h} G_{x_i} \times \int^{q} [q+1,x_j] \times [n, \sum_{i\in h\backslash\{j\}} x_i + q] \\
&\overset{⑧}{\cong} \int^{h} F_{h} \times \int^{x\in\widehat{\mathbb{B}}^{h}} \prod_{i\in h} G_{x_i} \times \sum_{j\in h} \int^{q} [q+1,x_j] \times [n, \sum_{i\in h\backslash\{j\}} x_i + q] \\
&\overset{⑨}{\cong} \int^{h} F_{h} \times \int^{x\in\widehat{\mathbb{B}}^{h}} \prod_{i\in h} G_{x_i} \times [n+1, \sum_{i\in h} x_i] \\
&\overset{⑩}{\cong} \partial(F \circ G)
\end{aligned}
$$

where we used the following facts at each step:

1. density formula on $p$, coends commute with products, Fubini theorem
2. Fubini theorem, coends commute with products
3. density formula on $F_{m+1}$, density formula on $G_{q+1}$, Cartesian product is symmetric in $Set$ up to isomorphism, coend commute with products
4. Fubini theorem, coends commute with products
5. property $1$ from above lemma
6. coends commute with coproducts, density formula on $m$
7. renaming of $p$ into $x_j$, Cartesian product is symmetric up to isomorphism
8. coproducts commute with coends, strength between $\times$ and $+$

9. property 2 from above lemma
10. density formula and definition

□

*Proof of proposition 3.2.11.* Take a map $f : X^{\cdot k} \to X$ with coherence axiom and consider:

$$\mathcal{Y}(k) = \mathcal{Y}(1+1+\cdots+1) \xrightarrow{\cong} \mathcal{Y}(1)^{\cdot k} \xrightarrow{e^{\cdot k}} X^{\cdot k} \xrightarrow{f} X$$

where the isomorphism is given by $\alpha$ from the monoid coherence natural isomorphisms. This gives a natural transformation $\widetilde{f} : \mathcal{Y}(k) \to X$ and by Yoneda lemma an element $c \overset{def}{=} \widetilde{f}_k(id_k) \in X(k)$.

Take an element $c \in X(k)$. We have by the universal property of the coend a wedge $inj_k$ so that we can consider:

$$X_k \times X^{\cdot k} \overset{inj_k}{\to} \int^k X_k \times X^{\cdot k} \overset{m}{\to} X$$

This gives a map $\mu_k : X_k \times X^{\cdot k} \to X$ (dinatural in $k$) and $\mu_k(c, -) : X^{\cdot k} \to X$ is the desired map.

This map satisfy the coherence law because it is a particular case of the associativity of $m$. We need to show both process are inverse to each other.

- $c \overset{?}{=} (\mu_k(c,-)e^{\cdot k}\alpha)_k(id_k)$

$$\begin{aligned} id_k &\overset{\alpha}{\longmapsto} [1\ldots 1, id_1 \ldots id_1, id_k] \\ &\overset{e^{\cdot k}}{\longmapsto} [1\ldots 1, e(id_1)\ldots e(id_1), id_k] \\ &\overset{\mu_k}{\longmapsto} m_k\Big([k, c[1\ldots 1, e(id_1)\ldots e(id_1), id_k]]\Big) \end{aligned}$$

  However $m(X \circ e) = \rho$ so in particular

$$\begin{aligned} m_k\Big([k, c[1\ldots 1, e(id_1)\ldots e(id_1), id_k]]\Big) &= m_k(X\circ e)_k\Big([k, c, [1\ldots 1, id_1 \ldots id_1, id_k]]\Big) \\ &= \rho_k\Big([k, c, [1\ldots 1, id_1 \ldots id_1, id_k]]\Big) \\ &= c \end{aligned}$$

  where we used the definition of $\rho$.

- $f \overset{?}{=} \mu_k((fe^{\cdot k}\alpha)_k(id_k), -)$

  Notice first that $(fe^{\cdot k}\alpha)_k(id_k) = f\Big([k, [1\ldots 1, e(id_1)\ldots e(id_1), id_k]]\Big)$.

We have for all $j, x_1, \dots, x_k \in \mathbb{B}$, $n \overset{def}{=} \sum_{i \in k} x_i$ and $p_i \in X_{x_i}$:

$$
\begin{aligned}
& m_n\Big(\Big[j, f\big([k, [1 \dots 1, e(id_1) \dots e(id_1), id_k]]\big), [x_1 \dots x_k, p_1 \dots p_k, g]\Big]\Big) \\
\overset{1}{=}\ & (m_n(f \circ X)_n)\Big(\Big[j, \big[k, [1 \dots 1, e(id_1) \dots e(id_1), id_k]\big], [x_1 \dots x_k, p_1 \dots p_k, g]\Big]\Big) \\
\overset{2}{=}\ & f_n(m^{\cdot k})_n \beta_n\Big([j, \big[k, [1 \dots 1, e(id_1) \dots e(id_1), id_k]\big], \big[x_1 \dots x_k, p_1 \dots p_k, g]\big]\Big) \\
\overset{3}{=}\ & f_n(m^{\cdot k})_n\Big(\Big[x_1 \dots x_k, \quad [1, e(id_1), [x_1, p_1, id_{x_1}]], \\
& \qquad \vdots \\
& \qquad , [1, e(id_1), [x_k, p_k, id_{x_k}]], g\Big]\Big) \\
\overset{4}{=}\ & f_j\Big(\Big[x_1 \dots x_k, \quad m(e \circ X)_{x_1}\big([1, id_1, [x_1, p_1, id_{x_1}]]\big), \\
& \qquad \vdots \\
& \qquad , m(e \circ X)_{x_k}\big([1, id_1, [x_k, p_k, id_{x_k}]]\big), g\Big]\Big) \\
\overset{5}{=}\ & f_n\Big(\Big[x_1 \dots x_k, \quad \lambda_{x_1}^{-1}\big([1, id_1, [x_1, p_1, id_{x_1}]]\big), \\
& \qquad \vdots \\
& \qquad , \lambda_{x_k}^{-1}\big([1, id_1, [x_k, p_k, id_{x_k}]]\big), g\Big]\Big) \\
\overset{6}{=}\ & f_n([x_1 \dots x_k, p_1 \dots x_k, g])
\end{aligned}
$$

as desired, where we used the following:

1. definition of $(f \circ X)$
2. coherence axiom of $f$, $\beta$ denotes the strength for $\cdot$ on $\circ$
3. definition of $\beta$
4. definition of $m^{\cdot k}$ and of $e \circ X$
5. coherence axiom from $X$ being a monoid
6. definition of $\lambda$

$\square$

*Proof of theorem 3.2.12.* (1) $\Leftarrow$ (2).

$$
\begin{tikzcd}
\mathcal{Y}(k) \arrow[r, "e^{\cdot k}"] \arrow[dr, "\hat{c}"'] & X^{\cdot k} \arrow[d, "f"] \arrow[dr, "f'h^{\cdot k}"] & \\
& X \arrow[r, "h"] & X'
\end{tikzcd}
$$

The above diagram commutes by definition of $\hat{c}$ and by hypothesis. We read it in $2$ ways:

$$\begin{aligned} id_k &\mapsto \hat{c}(id_k) \stackrel{def}{=} c \\ &\mapsto h_k(c) \end{aligned}$$

$$\begin{aligned} id_k &\mapsto e^{\cdot k}(id_k) \\ &\mapsto f' h^{\cdot k} e^{\cdot k}(id_k) \\ &= f'(he)^{\cdot k}(id_k) \\ &= f' e'^{\cdot k}(id_k) \\ &\stackrel{def}{=} \hat{c}'(id_k) \\ &\stackrel{def}{=} c' \end{aligned}$$

where we used a property of $\cdot$ and the definition of $h$. This gives us $h_k(c) = c'$.

$(1) \Rightarrow (2)$. Recall $f = \mu_k(c, -)$ and $f' = \mu'_k(c', -)$.

The commutativity of the diagram amounts to

$$m'_k\Big(\big[k, c', [x_1 \dots x_k, h_{x_1}(p_1) \dots h_{x_k}(p_k), f]\big]\Big) = h_k m_k\Big(\big[k, c, [x_1 \dots x_k, p_1 \dots p_k, f]\big]\Big)$$

Because $h$ is a monoid homomorphism this amounts to

$$m'_k\Big(\big[k, c', [x_1 \dots x_k, h_{x_1}(p_1) \dots h_{x_k}(p_k), f]\big]\Big) = m'_k\Big(\big[k, h_k(c), [x_1 \dots x_k, h_{x_1}(p_1) \dots h_{x_k}(p_k), f]\big]\Big)$$

which is true by hypothesis. $\square$

# Bibliography


[1] S. Awodey. *Category Theory*. Oxford Logic Guides, 2006.

[2] M. Fiore. "Second-Order and Dependently-Sorted Abstract Syntax". In: LICS'08; IEEE Computer Society Press (2008).

[3] M. Fiore and C.K. Hur. "On the construction of free algebras for equational systems". In: Theor. Comput. Sci. 410, 1704–1729 (2009).

[4] M. Fiore and C.-K. Hur. "Term equational Systems and Logics". In: MFPS XXIV. ENTCS, vol. 218, pp. 171–192 (2008).

[5] M. Fiore, G. Plotkin, and D. Turi. "Abstract Syntax and Variable Binding". In: LICS'99, IEEE Computer Society Press (1999).

[6] M. Fiore et al. "The Cartesian closed bicategory of generalised species of structures". In: Journal of the London Mathematical Society, vol.77 (2008).

[7] V. Ginzburg and M. Kapranov. "Koszul Duality for Operads". In: Duke Math. J. Volume 76 (1994).

[8] G.B. Im and G.M. Kelly. "A universal property of the convolution monoidal structure". In: Journal of Pure and Applied Algebra, volume 43 (1986).

[9] A. Joyal. "Foncteurs analytiques et espèces de structures". In: volume 1234 of Lecture Notes in Mathematics (1987).

[10] A. Joyal. "Une théorie combinatoire des séries formelles". In: Advances in Mathematics, Vol. 42 (1981).

[11] G.M. Kelly. "On the Operads of J.P. May". In: Unpublished manuscript, 1972. (Reprints in Theory and Applications of Categories, No. 13 (2005).

[12] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics 5. Springer-Verlag, 1971.

[13] F. Loregian. "This is the (co)end, my only (co)friend". In: arxiv:1501.02503v2 (2015).

[14] J.P. May. *The Geometry of Iterated loop Spaces*. Lecture Notes in Mathematics. Springer-Verlag, 1972.

[15] J. Power and E. Robinson. "Premonoidal Categories and Notions of Computation". In: Mathematical Structures in Computer Science archive, Volume 7 Issue 5 (1997).

[16] M. Tanaka. "Abstract Syntax and Variable Binding for Linear Binders". In: Mathematical Foundations of Computer Science 00, Lecture Notes in Comput. Sci. 1893 (2000).