A universal property in quantum theory

Mathieu Huot

École Normale Supérieure Paris-Saclay

Context

Situation

- Quantum as a resource is hard to manipulate
- Need for approximation of gates and fault tolerance
- Need for quantum programming languages and a high level picture for optimisations
- Staton (2015) : first equationnal theory for which Von Neumann's model is complete

Goals

- theoretical investigation : understand, explain and simplify the theory Staton proposed
- justification of Von Neumann's model as an extension of the standard model of pure quantum mechanics
- toward better languages, management of resources, combining with usual algebraic effects

Results

- Von Neumann's model of full quantum mechanics (QM) is the simplest possible that can interpret pure QM, allows discarding and measuring , and quotients by global phase
- Categorical framework is proposed to formalize the previous intuition
- Additionnal proofs in two settings of enriched category theory: metric, topological.

Reversible quantum circuit: Pure QM

The diagram is to be read from left to right.

- horizontal wire: qubit
- box: unitary transformation
- vertical wire: control operator
- meter device: measurement
- double wire: bit

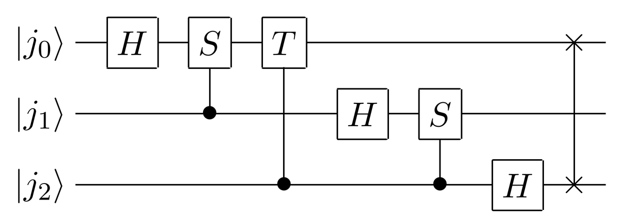


Fig. 1: Quantum fourrier transform on three qubits

Adding discarding: Measurement

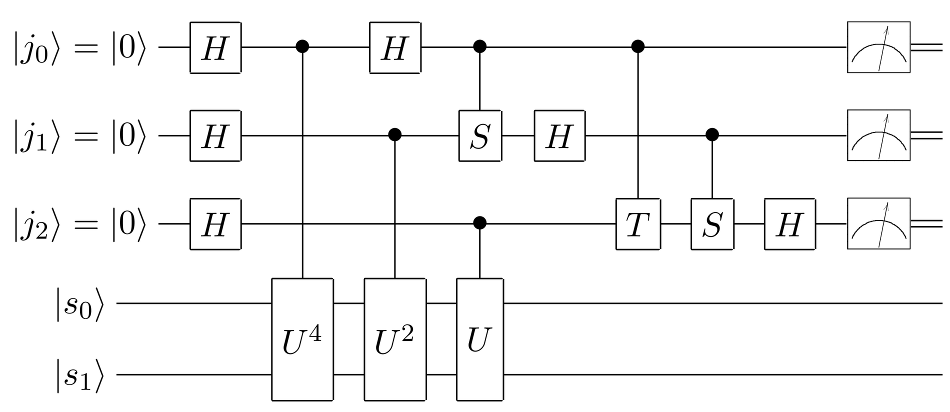


Fig. 2: Three-qubit phase estimation circuit with QFT and controlled-U

Prog. language: Staton (2015)

In continuation passing style:

$$\underbrace{\text{new}\big(a.}_{\text{qubit allocation}}\ \underbrace{\text{measure}\big(a}_{\text{branching depending on result}}\ ,\ \underbrace{x(a)}_{\text{continuation}}\ ,\underbrace{\text{apply}_X(a,}_{\text{apply unitary } X} x(a)\big)\big)$$

With equations, such as:

- $\text{new}\big(a.\text{measure}(a,x,y)\big) = x$: qubits are initialized to $|0\rangle$
- $\text{new}(a.\text{new}(b.x(a,b))) = \text{new}(b.\text{new}(a.x(a,b)))$

Pure quantum mechanics

Physics

- $|\psi\rangle \in \mathbb{C}^{2^n}$: state vector for n qubits
- $U : \mathbb{C}^{2^n} \to \mathbb{C}^{2^n}$: unitary (reversible) transformation
- $\mathbb{C}^{2^n} \otimes \mathbb{C}^{2^m} = \mathbb{C}^{2^{n+m}}$: combination of systems
- $U \otimes V$: action on subsystems by Kronecker product of unitaries
- couple the system with a prepared state $|\psi_0\rangle$

Category Theory

- objects are $n \in \mathbb{N}$, representing the state space $\mathbb{C}^{2^n}$
- invertible morphism $f_U : n \to n$ for every unitary U
- symmetric monoidal tensor product $\otimes$ given by:
$$\begin{cases} n \otimes m & := n + m \\ f_U \otimes f_V & := f_{U \otimes V} \end{cases}$$
- tensor with a morphism $f_{|\psi_0\rangle}$: $1 \to n$ thought of as a point in n

Quantum channels (Von Neumann)

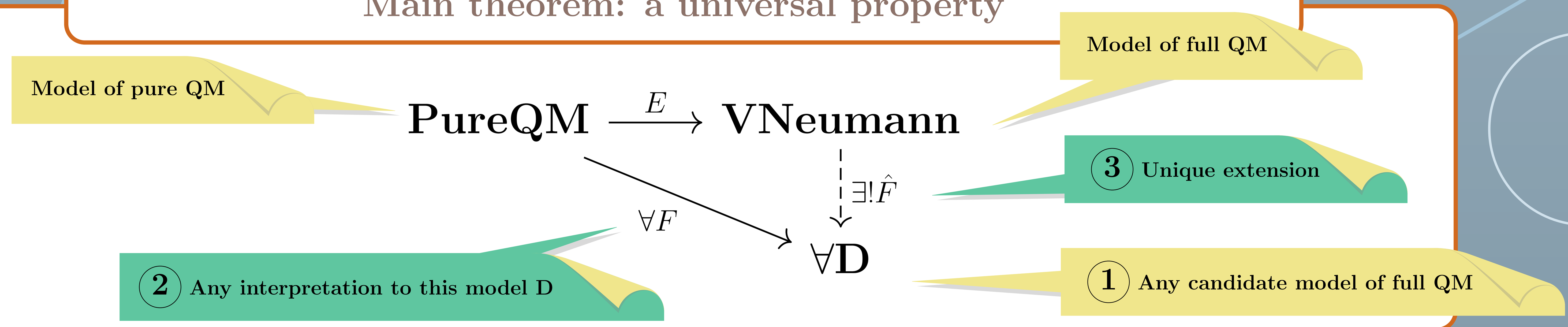
Physics

- $|\psi\rangle\langle\psi| := \langle\psi|^* \otimes \langle\psi| \in \mathcal{M}_{2^n}(\mathbb{C})$ pure state: density matrix
- $ad_U : \rho \mapsto U\rho U^*$: adjoint super-operator
- combination of systems by the tensor product $\otimes$
- no global phase: $|\psi\rangle = e^{i\theta}|\phi\rangle \Rightarrow |\psi\rangle\langle\psi| = |\phi\rangle\langle\phi|$
- allows discarding: measure ignoring the outcome, performed by the Trace operator

Category Theory

- objects are $n \in \mathbb{N}$, representing $\mathcal{M}_{2^n}(\mathbb{C})$
- a morphism $f : n \to m$ for each quantum channel $\mathcal{M}_{2^n}(\mathbb{C}) \to \mathcal{M}_{2^m}(\mathbb{C})$
- symmetric monoidal tensor product $\otimes$ given by:
$$\begin{cases} n \otimes m & := n + m \\ f \otimes g(a \otimes b) & := f(a) \otimes g(b) \text{ Kronecker product} \end{cases}$$
- interpretation E : **PureQM** $\to$ **VNeumann** functor preserving the monoidal structure

Main theorem: a universal property



Extensions

Enriched category theory in two settings:

- topological spaces
- metric spaces

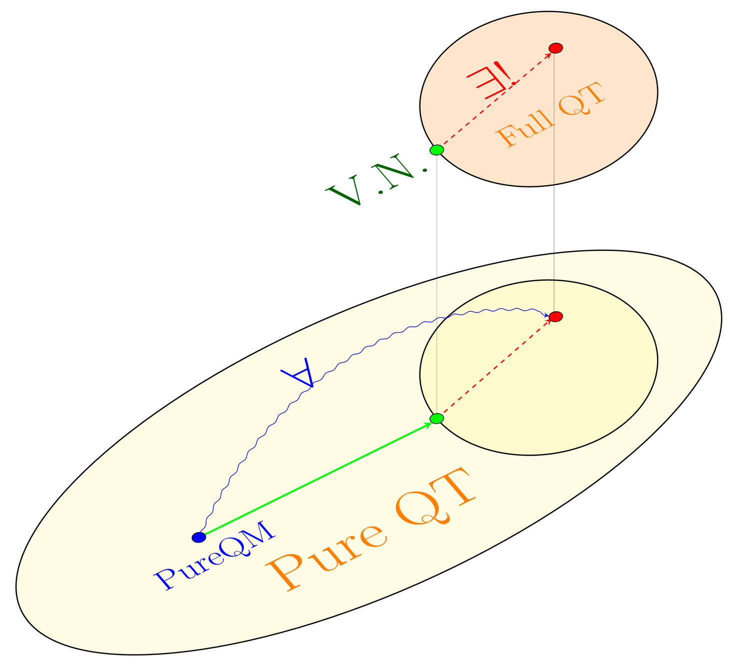
It means that the homset $[n, m]$ is a set with extra structure (a topology or a metric) and that the functors and natural transformations are required to preserve this extra structure.

A general setting, bipermutative categories, to consider more carefully discarding, measurement and entanglement for general quantum theories.

In each case we have a similar universal property in the form of a triangle.

Summary

Intituively, Von Neumann model of QM is the simplest model (V.N.) of a full quantum theory (QT) that faithfully interprets pure QM.
Every other model (in red) of a full QT with any interpretation to this model (in blue) factors uniquely through VN's model. Existence represents faithfulness of the interpretation and uniqueness the simplicity of the model.
In addition, this is also true in topological and metric quantum theories, and in the reffined version of bipermutative categories.



References

[1] Mathieu Huot and Sam Staton. "Universal Properties in Quantum Theory". In: (2018).

[2] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*. Cambridge university press, 2010.

[3] Sam Staton. "Algebraic effects, linearity, and quantum programming languages". In: *ACM SIGPLAN Notices*. Vol. 50. 1. ACM. 2015, pp. 395–406.

Acknowledgement

I would like to thank Younesse Kaddar, Ohad Kammar, Sean Moss, Sam Speight, Dario Stein, and my supervisor Sam Staton for a lot of useful discussions and great work.