A.R.P.E. Report

Quantum channels as a categorical completion

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August 22, 2018

This is the report concluding a ten months internship in the Computer Science department of Oxford University. This report has most of its content taken from a draft of article coauthored with Sam Staton to be submitted to LICS 2019. It has been entirely written by me.

We investigate the connection between pure and full quantum mechanics as used in quantum information theory. We propose a framework to consider models of quantum theories. We prove that the category of finite dimentional $\mathbb{C}^*$-algebras and unital $*$-homomorphisms is a completion of the category of finite dimentional vector spaces and unitaries. We further prove that the category of completely positive trace preserving (CPTP) maps between finite dimentional $\mathbb{C}^*$-algebras has a similar universal property but on the category of finite dimentional vector spaces and isometries. We extend those results to the enriched setting over topological spaces and also over metric spaces.

1 Introduction

Von Neuman's model of quantum theory accounts for full quantum mechanics (QM) in the sense that it incorporates a notion of mixedness which has no physicial reality but rather represents a state of knowledge of the observer. Uncertainty occurs after measurement and the model provides a convenient and concise way to deal with QM in such a setting. The model is quite fiddly as it uses super-operators and one would hope to find a simpler setting to do that.

Super-operators are also equipped with a metric. The metric is of primary importance in quantum computation to deal with fault tolerent programs and approximation of theoretical unitaries by physical gates.

We introduce a general algebraic framework concerned with models of pure and full quantum theories (QT). We use it to prove that Von Neuman's formalism is the simplest model that extends the usual model of pure QM and which allows measurement and tracing-out, and hence a notion of mixed states. We also propose a topological and metric versions of the framework and prove that our main result extends to that case. Thus, informally:

Theorem 1.1.
- *Von Neuman's model is the simplest one that extends the standard model of pure QM, quotients by global phase, admits discarding and measurement.*
- *In addition, the metric and the topology on super-operator are* canonically *given by the pure model of QM.*

Quantum computation

Quantum computation takes place in the setting of finite dimensional QM and we thus restrict ourselves to this setting. A system is given by the state vector ψ, which is a norm 1 vector in $\mathbb{C}^n$ for a certain n. A basic system consists of 1 qubit and is given by a norm 1 vector in $\mathbb{C}^2$. More generally a system of n-qubits is given by a norm 1 vector in $\mathbb{C}^{2^n}$. Operations on qubits are given by unitary transformations $U : \mathbb{C}^n \to \mathbb{C}^n$. The union of two systems $\phi \in \mathbb{C}^n, \psi \in \mathbb{C}^m$ is given by their tensor product $\phi \otimes \psi \in \mathbb{C}^n \otimes \mathbb{C}^m \cong \mathbb{C}^{n \times m}$. Transformations U, V on two subsystems act on the total system by their tensor product $U \otimes V$. It means U, V act on each subsystem but as the two sub-systems might be entangled, this might have a non trivial action on the global system. A system ψ can be coupled with an ancilla. An ancilla is a fixed prepared state ϕ_0. Ancillas play an important role and are crucial to quantum computation. This represents the model of pure – reversible – quantum computation and there is no measurement yet.

An example of pure system is given by the following diagram, to be read from left to right. The ket notation $|\psi\rangle$ is used for vectors. In this case each vector is a qubit so each wire represents the path of the qubit and the transformations applied to it. Several wires represent tensoring systems. A square represents a unitary transformation on a qubit, and the dot with a vertical line represents control.
Control is a way of entangling qubits. If a measurement is performed on the qubit with the dot, the result will tell whether the unitary will be applied on the other qubit.

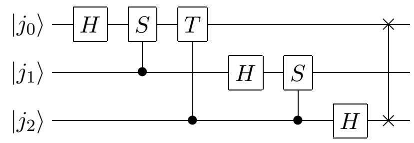


Figure 1: Quantum Fourier transform on three qubits

There are several equivalent ways to add measurement in the model. For instance one might add a set of projective operators $\{P_i\}$. Performing a measurement heavily perturbs the qubit and introduces uncertainty. For example, given projective operators P_0, P_1, if we perform a measurement of the first qubit on the system $\frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big)$, we get the mixed state $\left\{\frac{1}{2}\right\}|00\rangle + \left\{\frac{1}{2}\right\}|11\rangle$ which means we have half a chance of measuring $|0\rangle$ and having the system $|00\rangle$ and half a chance of measuring $|1\rangle$ and having the system $|11\rangle$. This has several drawbacks.

However, it seems *ad hoc* as we formally add some probabilities in curly braces. In addition, there is still a redundancy due to global phase. Whenever there is a real θ such that two states verify $\psi = e^{i\theta}\phi$, we say those states differ by a global phase θ. It is physically impossible to distinguish those two states as they will give the same probablities in every measurement basis. Instead, the model usually considered for full QM is the one of Von Neumann.

Von Neuman's model of quantum mechanics

Again, we restrict ourselves to finite dimension which is the setting for quantum computation. A pure state $\psi \in \mathbb{C}^n$ is interpreted as an $n \times n$ complex matrix $\psi\psi^* \in \mathcal{M}_n(\mathbb{C})$, where $(-)^*$ is the transpose conjugate. This matrix is a rank 1 positive matrix. More generally, a *density matrix* ρ is a positive matrix of trace 1 a represents a state (pure or mixed). Its rank k represents its degree of mixedness and every density matrix ρ of rank k can be written as a convex sum of (at least) k pure states $\rho = \sum_{1 \leq i \leq M} \lambda_i \psi_i \psi_i^*$. It represents the fact that the system is in the pure state $\psi_i\psi_i^*$ with probability λ_i.

Reversible transformations of states ρ are given by super-operators $ad_U : M \mapsto UMU^*$ where

U is a unitary matrix. It sends for instance the pure state $\psi\psi^*$ to $U\psi\psi^*U^* = (U\psi)(U\psi)^*$. They are called *super-operators* because they act on matrices which are themselves linear operators. The union of two systems is again given by the tensor product. It's given on matrices by the Kronecker product and on super-operators by the point-wise Kronecker product. Note that the global phase conveniently disappears in the process $\psi \mapsto \psi\psi^*$.

Measurement in the canonical base of a qubit is given by the projective measurement $\varphi : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_1(\mathbb{C}) \oplus \mathcal{M}_1(\mathbb{C})$ and sends the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to (a, d). It means that 0 is obtained with probability a, and 1 with probability d $(= 1 - a)$. It is also possible to just *discard* a qubit, in which case it will get eventually measured but we don't get any information on the measurement outcome. This is performed by the trace operator Tr and hence the operator $Tr_n \otimes Id_m$ – usually called partial trace – discards part of the system.

More generally, a valid transformation is a linear map T that sends density matrices to density matrices. It means T must preserve positive elements and trace. In addition, T might act on a substem of a bigger system via $T \otimes Id$ and so $T \otimes Id$ should also send density matrices to density matrices, and in particular preserve positive elements. A map T such that for all n the map $T \otimes Id_n$ preserves positive elements is called *completely positive*. A valid transformation is thereby a completely positve trace preserving map (CPTP). CPTP maps are also often called *quantum channels*.

An example is given by the diagram below. The meter device is measurement and double wires carry classical bits. The three first qubits are ancillas which are being measured (and discarded) at the end of the computation.

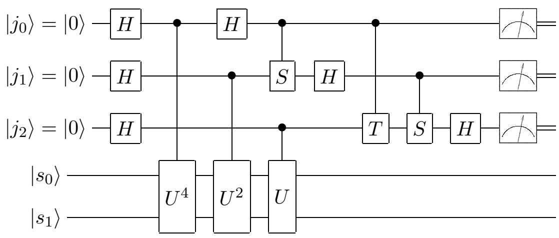


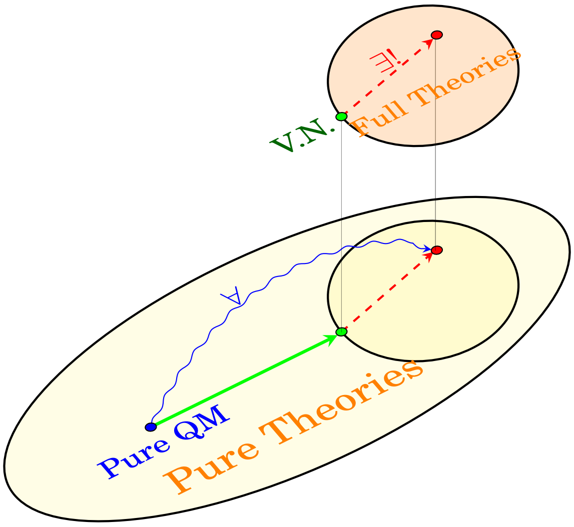
Figure 2: Three-qubit phase estimation circuit with QFT and controlled-U

The quantum channel model is convenient in many ways but is also quite fiddly, augments the dimension of the state space, and may seem *ad hoc* as well. In particular, it's not clear upfront whether every CPTP map should be allowed and it requires some difficult theorems to have simpler characterisations of CPTP maps.

Von Neuman's model is canonical

The main result of the paper is to prove that Von Neuman's model (V.N.) is canonical in the sense of category theory.

The general framework can be seen with the picture below which is to be read as follows. A bullet is a model of a quantum theory and an arrow is an interpretation/translation of a model to another. A model here at least interprets a pure QT. Some of the models are richer and interpret a full QT. The canonical completion of a model A of a pure QT into a model of a full QT is the closest B model of full QT that interprets A. For instance, the completion of the blue bullet is given by the green bullet and the green arrow gives the interpretation. We want the canonical completion to have the following universal property. Each interpretation (blue arrow) of the blue model into a model of a full QT (red bullet) can be uniquely lifted to an interpretation from the completion to the red model. The existence of a lifting means that the structure added to the blue model to give the green one does not crash information present in the blue model. The uniqueness means that we added the minimal information to the blue model to make it a model of a full QT. In this sense the completion of a model of a pure QT is the closest and simplest model of a full QT that encompasses the current model.



Presentation of the paper

In section §2 we introduce bipermutative categories and claim they are good general models for quantum theories. Augmenting a theory is performed by a completion, which we exemplify by proving that isometries are a completion of unitaries. Next, in section §3 we show that unital $*$-homomorphisms are a completion of unitaries, which is the key midpoint to our main theorem. In section §4 we expose our main theorem that CPTP are a completion of isometries and we extend it to both topological and metric enriched settings in section §5. We conclude with some related work and an overall summary (§6).

2 Bipermutative categories as models of QT

The standard models of QM can be nicely represented using categories and hence by a model of a quantum theory we mean a category with extra structure, where an object represents a state space and a morphism a valid transformation.

Intuitively, the structure required consist of the following. To consider the union of systems, the category is asked to be symmetric monoidal for a certain tensor product $\otimes$. Quantum entanglement occurs because of unitaries of the form $U \oplus V$ – the simplest example being the controlled-not operator $cX = (Id_2 \oplus X)$ – so a second symmetric tensor product $\oplus$ is required. This tensor product is more subtle because it does not have a direct physical interpretation but it is in a way the source of entanglement. Entanglement needs a way to be computed, and by mimicking the standard model we ask for a distributive law $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$. A way to understand this morphism is as a way to propagate information – given by $\oplus$ from a subsystem to the whole system. In pure QM $\oplus$ is entanglement and in full QM $\oplus$ is rather the uncertainty of the outcome triggered by the perturbation of entanglement caused by measurement.

A model for a full quantum theory should also admit discarding. We model discarding by asking the unit of the tensor product to be a terminal object. Measurement is more fiddly as it's a process of converting information. It converts information on the pure system to information (or lack of information) for the observer. In other terms we ask for a map $\phi : F(A \oplus B) \to F(A) \oplus F(B)$, where F embeds pure systems into full QM.

An important categorical ingredient in this paper is the notion of completion of a category and by this we mean adding structure in a universal way. We exemplify it in this section by showing that isometries are a simple completion of unitaries.

2.1 General setting

Definition 2.1. *A* rig category *is a category with two symmetric monoidal structures,* $(\oplus, \gamma, N)$ *and* $(\otimes, \gamma', I)$*, such that one distributes over the other, that is, there are isomorphisms* $A \otimes (B \oplus C) \to (A \oplus B) \otimes (A \oplus C)$ *and* $(A \oplus B) \otimes C \to (A \oplus C) \otimes (B \oplus C)$ *satisfying some coherence conditions.*

Example 2.2. *Any rig A (ring except inverses for addition) can be seen as a bipermutative category with one object $*$ and one morphism a for each object $a \in A$.*

The coherence conditions are a bit cumbersome, but they are simplified in many cases. We can often choose a strict monoidal structure (so associativity and unit isomorphisms are actually identity morphisms between equal objects) and we can choose one of the distributivity isomorphisms to be an identity morphism. This leads to the definition of bipermutative category [44]:

Definition 2.3. *A* bipermutative category *comprises a category $\mathcal{C}$ with two symmetric strict monoidal structures,* $(\oplus, \gamma, N)$ *and* $(\otimes, \gamma', I)$ *together with natural isomorphisms* $\lambda^* : N \otimes A \cong N$*,* $\rho^* : A \otimes N \cong N$*,* $\delta : (A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$*,* $\delta^\sharp : A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$*, with certain coherence conditions.*

Definition 2.4. *A* strict *bipermutative functor is a functor between bipermutative categories that strictly preserves all the structure:* $F(N) = N$*,* $F(I) = I$*,* $F(A \otimes B) = F(A) \otimes F(B)$*,* $F(A \oplus B) = F(A) \oplus F(B)$*,* $F(\gamma) = \gamma$*,* $F(\gamma') = \gamma'$*.*

Definition 2.5. *A* $\oplus$-lax *bipermutative functor is a functor* $F : \mathcal{A} \to \mathcal{B}$ *between bipermutative categories such that* $F(N) = N$*,* $F(I) = I$*,* $F(A \otimes B) = F(A) \otimes F(B)$*,* $F(\gamma') = \gamma'$ *together with a natural transformation* $\varphi : F(A) \oplus F(B) \to F(A \oplus B)$ *such that certain coherence diagrams commute.*

We have a coherence theorem for rig categories from (Laplaza [39]) which implies coherence for bipermutative categories.

Proposition 2.6. *Coherence for bipermutative categories.*

We will consider categories whose objects are bipermutative categories and whose morphisms are either lax or strict bipermutative functors. We will sometimes restrict to bipermutative categories where N is an initial object, or I is a terminal object, or $\oplus$ is a coproduct; in the latter case δ, $\delta^\sharp$ and γ should be the canonical morphisms.

Notation 2.7. *We write* $\mathcal{R}(\underset{s}{\otimes}\,\underset{s}{I}\,\underset{s}{\oplus}\,\underset{s}{N})$ *for the category of bipermutative categories and strict bipermutative morphisms, and* $\mathcal{R}(\underset{s}{\otimes}\,\underset{s}{I}\,\underset{l}{+}\,\underset{s}{0})$ *for the category of bipermutative categories with coproducts and $\oplus$-lax bipermutative morphisms, and so on.*

2.2 Models of QM

Definition 2.8. *A* unitary *U is an $n \times n$ complex matrix such that $U^*U = UU^* = I$, where U^* is the conjugate transpose. The bipermutative category of unitaries is formed as follows. The objects are natural numbers (including zero). There is a morphism $m \to n$ for each $n \times n$ unitary.*

- *Composition of morphisms is matrix multiplication*
- *On objects, $\oplus$ is addition of numbers. On morphisms it leads to control gates. Given unitaries $U : n \to n$ and $V : m \to m$, we let $U \oplus V : n \oplus m \to n \oplus m$ be the block diagonal matrix $\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$. The unit N is the number zero.*
- *On object $\otimes$ is multiplication of numbers. On morphisms, given unitaries $U : n \to n$ and $V : m \to m$, we let $U \otimes V : n \otimes m \to n \otimes m$ be Kronecker product of matrices, with $(U \otimes V)_{in+k,jn+l} = U_{i,j}V_{k,l}$. The unit I is the number 1. Intuitively, V acts on the subsystem n of nm without perturbing the substem m.*

Definition 2.9. *An* isometry *is a linear map $\mathbb{C}^m \to \mathbb{C}^n$ which preserves the inner product metric. In other words, an isometry is an $n \times m$ complex matrix V such that $V^*V = I$. Note that necessarily $m \leq n$ and $m = n$ precisely when an isometry is unitary. We form the bipermutative category of isometries in the very same way as the one for unitaries. The unit N is the number zero; it is an initial object, as witnessed by the empty matrices $¡_n : 0 \to n$.*

A positive element A in a $\mathbb{C}^*$-algebra is such that there exists a B such that $A = B^*B$, or equivalently if it is self-adjoint and its spectrum $\sigma(A)$ consists of non-negative real numbers.

A linear map $f : A \to B$ between two $\mathbb{C}^*$-algebras is positive if it maps positive elements to positive elements, *i.e.* $a \geq 0 \Rightarrow f(a) \geq 0$.

A linear map $f : A \to B$ between two $\mathbb{C}^*$-algebras is completely positive if for every k the map $id_k \otimes f : \mathcal{M}_k \otimes A \to \mathcal{M}_k \otimes B$ is positive.

Definition 2.10. *The bipermutative category of CPTP is defined as follows. Its objects are finite lists of natural numbers and a morphism $f : [n_1, \ldots, n_k] \to [m_1, \ldots, m_p]$ is a completely positive map $f : \bigoplus_i \mathcal{M}_{n_i}(\mathbb{C}) \to \bigoplus_i \mathcal{M}_{m_i}(\mathbb{C})$. For instance $\mathcal{M}_2(\mathbb{C})$ is the type of qubits and $\mathbb{C} \oplus \mathbb{C}$ is the type for bits.*

$\oplus$ is given on object by concatenation of lists and on morphisms $f : A \to B, g : A' \to B'$ by $(f \oplus g)(a, b) := \big(f(a), g(b)\big)$. $\oplus$ is a coproduct in this category. The empty list is the initial object. $\otimes$ is given on objects by $[n_1, \ldots, n_k] \otimes [m_1, \ldots, m_p] := [n_1m_1, \ldots, n_1m_p, \ldots, n_km_1, \ldots, n_km_p]$ and on morphisms $f : A \to [n], g : B \to [m]$ by $(f \otimes g) : A \otimes B \to [nm] : (a \otimes b) \mapsto f(a) \otimes f(b)$ where $\otimes$ is the Kronecker product. $[1]$ – to be thought as $\mathbb{C}$ – is the terminal object and the terminal map $!_n : [n]_to1$ is given by the trace operator. The map $!_n \otimes id_m : [nm] \to [m]$ is usually called the partial trace.

Definition 2.11. *There is a $\oplus$-colax-bipermutative functor $\mathbb{E}$:* **Isometry** $\to$ **CPTP** *which sends $n \to [n]$ and $V : m \to n$ to $ad_V : M \mapsto VMV^* : [m] \to [n]$. The colax $\oplus$-morphism $\varphi_{n,m} : \mathcal{M}_{n+m}(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{M}_m(\mathbb{C})$ is given by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto (A, D)$.*

Intuitively, $\varphi_{2^n,2^n} : \mathcal{M}_{2^{n+1}}(\mathbb{C}) \to (\mathbb{C}) \oplus \mathbb{C}) \otimes \mathcal{M}_{2^n}(\mathbb{C}))$ measures the first qubit of a system of $n + 1$ qubits, and the resulsting system of n qubits is A if 0 is measured, and D otherwise.

2.3 Completions

As said in the introduction, the main result of the paper concerns completions. To give a first example of completion, we show the category of isometries is the initial object completion of the category of unitaries, both seen as bipermutative categories.

The initial map ¡ is used to embed a system in bigger system, or equivalently to add a fixed ancilla to a system. The completion then reads as follows: the category of isometries is the simplest model for pure quantum mechanics with ancillas.

The following proposition is the key point is the proof of the completion:

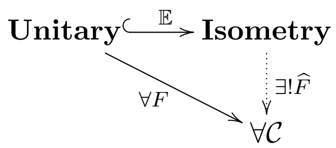
Proposition 2.12. *If $V : m \to n$ is an isometry then there is a unitary $U : n \to n$ such that $V = U(I_m \oplus \text{¡}_p)$, where $n = m + p$.*

Moreover, U is essentially unique: if $U_1, U_2 : n \to n$ are such that $U_1(I_m \oplus !_p) = U_2(I_m \oplus !_p)$, then there is a unique unitary $W : p \to p$ such that $U_2 = U_1(I_m \oplus W)$.

We can now state the theorem of the section:

Theorem 2.1. $\mathcal{R}(\overset{\otimes}{s}\,\overset{I}{s}\,\overset{\oplus}{s}\,\overset{0}{s}) \to \mathcal{R}(\overset{\otimes}{s}\,\overset{I}{s}\,\overset{\oplus}{s}\,\overset{\mathbb{O}}{s})$

The embedding exhibits the ringoid of finite dimensional Hilbert spaces and isometries **Isometry** *as the initial object completion of the ringoid of finite dimensional unitaries* **Unitary**:



where F is a morphism in $\mathcal{R}(\overset{\otimes}{s}\,\overset{I}{s}\,\overset{\oplus}{s}\,\overset{\mathbb{O}}{s})$ and $\widehat{F}$ a morphism in $\mathcal{R}(\overset{\otimes}{s}\,\overset{I}{s}\,\overset{\oplus}{s}\,\overset{0}{s})$.

3 Full QM without ancillas

As opposed to the category of isometries, unitaries are a model of pure QM without ancillas. So it is expected that the completion of unitaries will be a model of full QM without ancillas, a subcategory of CPTP. However, this category turns to be hard to work this out directly. We hence take a different approach and use duality. A known result on duality states the dual of the category of CPTP is the category of CP maps which are unital. It has a non-full subcategory of unital $*$-homomorphisms. We show the category of unital $*$-homomorphisms is a completion of the opposite of the category of unitaries and then obtain the main result of the section by duality.

3.1 Duality

$*$-homomorphisms are morphisms that respect the structure of $\mathbb{C}^*$-algebras. In finite dimension, $\mathbb{C}^*$-algebras are all of the form $\bigoplus_i \mathcal{M}_{n_i}(\mathbb{C})$, up to isomorphism. A $\mathbb{C}^*$-algebra has three operations : an addition $+$, a multiplication $.$ and an involution $(-)^*$. Here, $+$ is given by addition of matrices, $.$ by matrix multiplication and $(-)^*$ by the transpose conjugate. Every $\mathbb{C}^*$-algebra $\bigoplus_i \mathcal{M}_{n_i}(\mathbb{C})$ is unital. Its unit is given by $(Id_{n_1}, \dots, Id_{n_k})$ where Id_{n_i} is the identity matrix.

Definition 3.1. *A bounded linear operator $\pi : A \to B$ between $\mathbb{C}^*$-algebras is called a $*$-homomorphism if:*

- $\forall x, y \in A, \pi(x.y) = \pi(x).\pi(y)$
- $\forall x \in A, \pi(x^*) = \pi(x)^*$

It is unital if it preserves the unit of the $\mathbb{C}^*$-algebra.

Definition 3.2. *We have a bipermutative category* **Cstar** *of unital $*$-homomorphisms. Its objects are finite lists of natural numbers and a morphism $f : (n_1, \dots, n_k) \to (m_1, \dots, m_p)$ is a unital $*$-homomorphism $f : \bigoplus_i \mathcal{M}_{n_i} \to \bigoplus_j \mathcal{M}_{m_j}$.*
$\oplus, \otimes$ are given similarly that in the case of CPTP.

Similarly, we can define the category **CPU** of completely positive unital map, which contains **Cstar** has a non full subcategory. We have the following known result from Choi [17]:

Proposition 3.3. *(Choi) $f : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ is CP iff it admits an expression $f(A) = \sum_i V_i^* A V_i$ where V_i are $n \times m$ matrices.*

Corollary 3.4. $\mathbf{CPU}^{op} = \mathbf{CPTP}$.

Thanks to the work of Bratteli [11], we have an explicit characterisation of unital $*$-homomorphisms in finite dimension, given as follows:

Proposition 3.5. *(Bratteli) $f : \bigoplus_{1 \leq i \leq k} \mathcal{M}_{n_i}(\mathbb{C}) \to \mathcal{M}_p(\mathbb{C})$ is a $*$-homomorphism iff there exist a $p \times p$ unitary U and natural numbers $s_1, \ldots, s_k$ such that $\sum_{1 \leq i \leq k} n_i s_i = p$ and*

$$f(A_1, \ldots, A_k) = U(A_1 \otimes Id_{s_1} \oplus \ldots \oplus A_k \otimes Id_{s_k})U^*$$

However, this is true under the identification of $\mathcal{M}_n \oplus \mathcal{M}_m$ with the image of the inclusion $\varphi : \mathcal{M}_n \oplus \mathcal{M}_m \to \mathcal{M}_{n+m}$.

Notation 3.6.

- *$\phi : \bigoplus_i \mathcal{M}_{n_i}(\mathbb{C}) \to \mathcal{M}_{\sum_i n_i}(\mathbb{C})$ the canonical injection*
- *$\Delta_{s,n} : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_{n_i}(\mathbb{C})^{\oplus s}$ given by the universal property of the product, sends A to $(A, \ldots, A)$*
- *$ad_U : M \to U^* M U$*

Rephrasing the characterisation of Bratteli, we get:

Corollary 3.7. *Unital $*$-homomorphisms $\bigoplus_{1 \leq i \leq k} \mathcal{M}_{n_i}(\mathbb{C}) \to \mathcal{M}_p(\mathbb{C})$ are exactly morphisms of the form $ad_U \circ \phi \circ \bigoplus_i \Delta_{s_i, n_i}$ for some $p \times p$ unitary U and $\sum_i s_i n_i = p$.*

3.2 Completion of Unitary

The theorem rests upon a few lemmas:

Lemma 3.8. *$ad_U \circ \varphi \circ \Delta_s = ad_{U'} \circ \varphi \circ \Delta_s : \mathcal{M}_n \to \mathcal{M}_{ns}$ iff there exists an $s \times s$ unitary V such that $U \circ (V \otimes Id_n) = U'$.*

Proof. For the converse, notice that $\Delta_{s,n} = \Delta_{s,1} \otimes I_n$. Also note that $\varphi_{n,n} = \varphi_{1,1} \otimes Id_n$. Then for all $U : ns \to ns, V : s \to s$ we have, using the fact that 1 is the initial object:

$$\begin{aligned} ad_U \circ ad_{V \otimes Id_n} \circ \varphi \circ \Delta_{s,n} &= ad_U \circ (ad_V \otimes Id_n) \circ (\varphi \otimes Id_n) \circ (\Delta_{s,1} \otimes Id_n) \\ &= ad_U \circ \big((ad_V \circ \varphi \circ \Delta_{s,1}) \otimes Id_n\big) \\ &= ad_U \circ \big((\varphi \circ \Delta_{s,1}) \otimes Id_n\big) \\ &= ad_U \circ \varphi \circ \Delta_{s,n} \end{aligned}$$

We now prove the first part.Let $M := \sum_{0 \leq i < n} E_{j+im, j+im}$. Then for all $k \notin \{j+im \mid i = 0 \ldots n-1\}$ we have

$$0 = M_{k,k} = (UMU^*)_{k,k} = \sum_{0 \leq i < n} a_{k,j+im} m_{j,j} \bar{a}_{k,j+im}$$

hence $\sum_{0 \leq i < n} |a_{k,j+im}|^2 = 0$ so $\forall i \in \{0, \ldots, n-1\}$, $a_{k,j+im} = 0$.

- For arbitrary M and (i,j) where $m_{i,j}$ is not necessarily 0 we have:

$$m_{i,j} = \sum_{0 \leq k < n} a_{i,i+km} m_{i,j} \bar{a}_{j,j+km}$$

hence $\sum_{0 \leq k < n} a_{i,i+km} \bar{a}_{j,j+km} = 1$.

Using Cauchy-Schwarz inequality we get:

$$\begin{aligned} 1 &= \sum_{0 \leq k < n} a_{i,i+km}\bar{a}_{j,j+km} \\ &= \langle (a_{i,i}, \dots, a_{i,i+(n-1)m}), (a_{j,j}, \dots, a_{j,j+(n-1)m}) \rangle \\ &\leq \left\| (a_{i,i}, \dots, a_{i,i+(n-1)m}) \right\|^2 . \left\| (a_{j,j}, \dots, a_{j,j+(n-1)m}) \right\|^2 \\ &= 1 \end{aligned}$$

Again by Cauchy Schwarz theorem, there exists a complex number λ such that

$$(a_{i,i}, \dots, a_{i,i+(n-1)m}) = \lambda (a_{j,j}, \dots, a_{j,j+(n-1)m})$$

Replacing in the equality above leads to $1 = \sum_{0 \leq k < n} \bar{\lambda} |a_{i,i+km}|^2 = \bar{\lambda}$ hence $\lambda = 1$. Consider $V = (a_{1+im,1+jm})_{0 \leq i,j < n}$. It is a unitary matrix and it verifies $V \otimes I_m = U$. □

Lemma 3.9. *If $ad_U \circ \varphi \circ \bigoplus_i \Delta_{s_i,n_i} = \varphi \circ \bigoplus_i \Delta_{s_i,n_i}$, then $U = \bigoplus_i U_i$ for some $s_i n_i \times s_i n_i$ unitaries U_i.*

Proof. It suffices to show that $U = U_1 \oplus U_2$ with $U_1 : s_1 n_1 \to s_1 n_1$ and then conclude by induction on the size of I, the set of indices in the sum.

Let $M := \sum_{s_1 n_1 < j \leq p} E_{j,j}$. Then, for $1 \leq k \leq s_1 n_1$, we obtain:

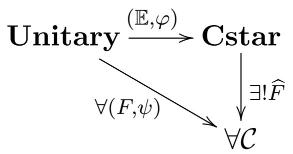
$$0 = M_{k,k} = \sum_{s_1 n_1 < j \leq p} a_{k,j} M_{j,j} \bar{a_{k,j}}$$

Hence for all $1 < k \leq s_1 n_1, s_1 n_1 < j \leq p$ we have $a_{k,j} = 0$.

As $U^* M U = M$, we get $U M U^* = M$ hence for all $1 < k \leq s_1 n_1, s_1 n_1 < j \leq p$, we have $a_{j,k} = 0$. This indeed shows that $U = U_1 \oplus U_2$, as desired. □

Theorem 3.1. $\mathcal{R}\left(\begin{smallmatrix} \otimes & 0 & \times & 1 \\ s & s & s & s \end{smallmatrix}\right) \to \mathcal{R}\left(\begin{smallmatrix} \otimes & I & \oplus & N \\ s & s & l & s \end{smallmatrix}\right)$

The functor exhibits the ringoid of finite dimensional C-algebras and unital *-homomorphisms* **Cstar** *as the $(0, s), (\times, l)$-completion of the groupoid of finite dimensional unitaries* **Unitary**. *This means we have the following diagram:*



where (F, ψ) is a morphism in $\mathcal{R}\left(\begin{smallmatrix} \otimes & I & \oplus & \mathbb{O} \\ s & s & l & s \end{smallmatrix}\right)$ and $\widehat{F}$ a morphism in $\mathcal{R}\left(\begin{smallmatrix} \otimes & 0 & \times & 1 \\ s & s & s & s \end{smallmatrix}\right)$.

Proof notes. Uniqueness. $\widehat{F}$ has to strictly preserve $\otimes, \oplus$ and make the triangle commute so $\widehat{F}$ must satisfy:

- $\widehat{F}[n_1, \dots, n_k] = \bigoplus_i F(n_i)$
- $\widehat{F}(f) = F(U) \circ \psi \circ \bigoplus_i \Delta_i$

Existence. We define $\widehat{F}$ is the unique possible way. $\widehat{F}$ is will be well defined by the key lemmas. $\widehat{F}$ is a functor is proven by long but straightforward rewriting. It's easy to show that $\widehat{F}$ preserves $\oplus$ and $\otimes$ on morphisms. □

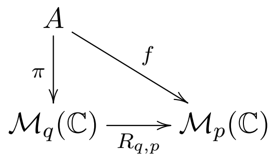
4 Quantum channels

We prove that **CPU** is a completion of $\mathbf{Isometry}^{op}$ and obtain our main result by duality: Von Neuman's model for full QM **CPTP** is a canonical completion of the usual model of pure QM with ancilla **Isometry**. The main technical known tool is Stinespring theorem which characterise CP maps.

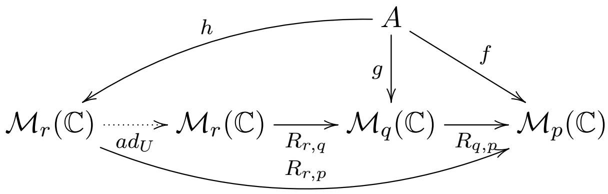
4.1 Stinespring theorem

Stinespring theorem is a very powerful tool even in infinite dimension. In finite dimension and in the case of CPU maps it can be stated as follows:

Theorem 4.1. *(Stinespring) Let p be a natural number. If $f : A \to \mathcal{M}_p(\mathbb{C})$ is a completely positive and unital map, then there is a natural number $q \geq p$ and a unital $*$-homomorphism π making the following diagram commute:*



where $R_{q,p}$ is the restriction to the first p rows and columns $A \mapsto A|_p$. Moreover, q can chosen to be the minimal such number: if $r \geq p$ and a $*$-homomorphism $h : A \to \mathcal{M}_r$ is such that $h(-)|_p = f(-)$ then $r \geq q$ and there is a unitary U such that $g(-) = (Uh(-)U^*)|_q$. In a diagram:



In other words, every CPU map can be written as a unital $*$-homomorphism followed by a restriction map, and possibly in a sort of minimal but non unique way. Written categorically, we get the following:

Corollary 4.1. *CPU maps $\bigoplus_{1 \leq i \leq k} \mathcal{M}_{n_i}(\mathbb{C}) \to \mathcal{M}_p(\mathbb{C})$ are exactly morphisms of the form $R_{q,p} \circ ad_U \circ \phi \circ \bigoplus_i \Delta_{s_i,n_i}$ for some $q \times q$ unitary U and $\sum_i s_i n_i = q \geq p$.*

Remark 4.2. *Recall that we had a functor $\mathbb{E} : \mathbf{Isometry}^{op} \to \mathbf{CPU}$. We have $R_{r,q} = \mathbb{E}(id_q \oplus !_{r-q})$.*

4.2 Completion of Isometry

Similarly to the main result of the previous section, we need a few key lemmas which are summed up in the proposition below:

Proposition 4.3. *$R_{m,p} \circ ad_U \circ \varphi \circ \bigoplus_{1 \leq i \leq k} \Delta_{s_i,n_i} = R_{m,p} \circ ad_{U'} \circ \varphi \circ \bigoplus_{1 \leq i \leq k} \Delta_{s_i,n_i} : \bigoplus_i \mathcal{M}_{n_i} \to \mathcal{M}_p$ iff there exist unitary matrices $P, Q_1, \ldots, Q_k$ such that*

$$(Id_{m-p} \oplus P) \circ U \circ \big((Q_1 \otimes Id_{n_1}) \oplus \cdots \oplus (Q_k \otimes Id_{n_k})\big) = U'$$

Proof notes. Suppose $f = f_{V_1^*} \circ \pi_2 = f_{V_2^*} \circ \pi_2$ for unital $*$-homomorphisms π_i. There exists a minimal dilation $f = f_U \circ \pi_1$. Using the corollary of the uniqueness in Stinespring theorem, there exists two isometries W_1, W_2 such that $f_{W_i} \circ \pi_1 = \pi_2$ and $W_i U = V_i$.

We now mimick the proof of the useful lemma and we get that $W_i = \bigoplus_j W_{i,j}$. Indeed, one can write $\pi_2 = f_T \circ \phi \circ \bigoplus_i \Delta_{s'_i}$ for some unitary T and $\pi_1 = f_Q \circ \phi \circ \bigoplus_i \Delta_{s_i}$ for some unitary Q. Multiplying by $f_{T^{-1}}$ shows that without loss of generality we can take π_2 to be $\phi \circ \bigoplus_i \Delta_{s'_i}$ and π_1 to be $\phi \circ \bigoplus_i \Delta_{s_i}$ if we change V into $T^{-1}VQ$. Then, just as in the proof of the useful lemma, we get that

$$m_{i,j} = \sum_{1\leq p\leq n} \sum_{1\leq k\leq n} v_{i,k} m_{k,p} \bar{v}_{j,p}$$

and taking M to be ones on the diagonal of the first block and 0 elsewhere gives that for all i not in the first block, $\sum_{1\leq p\leq end\ block1} v_{i,p}\bar{v}_{i,p} = 0$ hence $v_{i,p} = 0$ for all i not in the first block of indices, and for every p in the first block of indices. Doing this for the other blocks gives that V is of the form $V = \bigoplus_i V_i$.

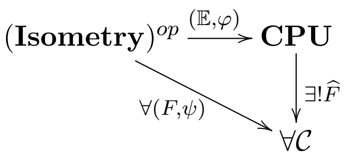
Now precomposing by the injection $\mathcal{M}_{n_j} \to \bigoplus_i \mathcal{M}_{n_i}$ and using the known result for quantum channels and the characterisation of isometries , $W_{i,j} = (P_{i,j} \otimes Id) \circ (Id \oplus \text{¡})$.

Putting everything together, there is a permutation $\widetilde{\gamma}$ independant of i such that $W_i = (P_{i,1} \otimes Id \oplus \cdots \oplus P_{i,k} \otimes Id) \circ \widetilde{\gamma} \circ (Id \oplus \text{¡})$. Hence

$$\begin{aligned}
&(P_{1,1}P_{2,1}^{-1} \otimes Id \oplus \cdots \oplus P_{1,k}P_{2,k}^{-1} \otimes Id) \circ V_2 \\
&= (P_{1,1}P_{2,1}^{-1} \otimes Id \oplus \cdots \oplus P_{1,k}P_{2,k}^{-1} \otimes Id) \circ W_2 U \\
&= (P_{1,1}P_{2,1}^{-1} \otimes Id \oplus \cdots \oplus P_{1,k}P_{2,k}^{-1} \otimes Id) \circ (P_{2,1} \otimes Id \oplus \cdots \oplus P_{2,k} \otimes Id) \circ \widetilde{\gamma} \circ (Id \oplus \text{¡}) \circ U \\
&= (P_{1,1} \otimes Id \oplus \cdots \oplus P_{1,k} \otimes Id) \circ \widetilde{\gamma} \circ (Id \oplus \text{¡}) \circ U \\
&= W_1 \circ U \\
&= V_1
\end{aligned}$$

□

Theorem 4.2. $\mathcal{R}(\begin{smallmatrix}\otimes & 0 & \times & 1\\ s & s & s & s\end{smallmatrix}) \to \mathcal{R}(\begin{smallmatrix}\otimes & I & \oplus & 1\\ s & s & l & s\end{smallmatrix})$
The functor exhibits the ringoid of finite dimensional C-algebras and completely positive unital maps* **CPU** *as the* $(0, s), (\times, l)$*-completion of the category of finite dimensional Hilbert spaces and co-isometries* $(\textbf{Isometry})^{op}$.



where (F,ψ) *is a morphism in* $\mathcal{R}(\begin{smallmatrix}\otimes & I & \oplus & 1\\ s & s & l & s\end{smallmatrix})$ *and* $\widehat{F}$ *a morphism in* $\mathcal{R}(\begin{smallmatrix}\otimes & 0 & \times & 1\\ s & s & s & s\end{smallmatrix})$.

Proof notes. Uniqueness is obvious from Stinespring theorem and the fact that $\widehat{F}$ has to be a functor and preserve $\oplus, \otimes$.
Existence: there is a unique obvious way to define $\widehat{F}$. $\widehat{F}$ is well defined on morphisms essentially by using the previous proposition. Note that it's important that we don't define $\widehat{F}$ on minimal dilations only. Again using the previous proposition, one can show that $\widehat{F}$ preserves composition. It is easy to show that $\widehat{F}$ preserves $\oplus$ and $\otimes$. □

By duality, we obtain our main result:

Corollary 4.4. $\mathcal{R}(\begin{smallmatrix}\otimes & 1 & + & 0\\ s & s & s & s\end{smallmatrix}) \to \mathcal{R}(\begin{smallmatrix}\otimes & I & \oplus & 0\\ s & s & l & s\end{smallmatrix})$

The functor exhibits **CPTP** *as the* $(1,s),(+,l)$*-completion of the category* **Isometry**.

$$\begin{array}{ccc}
\mathbf{Isometry} & \xrightarrow{(\mathbb{E},\varphi)} & \mathbf{CPTP} \\
& \searrow^{\forall(F,\psi)} & \downarrow{\exists!\widehat{F}} \\
& & \forall\mathcal{C}
\end{array}$$

where (F,ψ) *is a morphism in* $\mathcal{R}\left(\begin{smallmatrix}\otimes & 1 & \oplus & \mathbb{0}\\ s & s & l & s\end{smallmatrix}\right)$ *and* $\widehat{F}$ *a morphism in* $\mathcal{R}\left(\begin{smallmatrix}\otimes & 1 & + & 0\\ s & s & s & s\end{smallmatrix}\right)$.

Remark 4.5. *We have the following picture:*

$$\begin{array}{ccc}
\mathbf{Unitary} & \longrightarrow & \mathbf{Cstar} \\
\downarrow & & \color{red}{\downarrow} \\
\mathbf{Isometry} & \longrightarrow & \mathbf{CPTP}
\end{array}$$

Both horizontal arrows express the same completion, and the left vertical arrow is a different completion. The red arrow is then given by the universal property of **Cstar** *as a completion of* **Unitary**. *Using Theorem 2.1, this sheds a new light on Stinespring theorem. Stinespring theorem can now be understood as the lifting of the completion of unitaries into isometries. Intuitively, what makes the functor in red non full is the image by the horizontal completion of what is added by the left vertical one. Stinespring theorem makes this precise.*

5 As enriched categories

We now propose to go beyond our algebraic framework and consider the context of enriched category theory. Given a category $\mathcal{V}$, a $\mathcal{V}$-bipermutative category is a category $\mathcal{C}$ enriched over $\mathcal{V}$ with $\mathcal{V}$-enriched bifunctors $\oplus, \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\mathcal{V}$-enriched natural transformations γ, γ' such that the coherence conditions of bipermutative categories are satisfied. There is a category of $\mathcal{V}$-bipermutative categories and $\mathcal{V}$-bipermutative functors between them.

We prove an enriched version of the main theorem from the previous section in two settings. The first one is $\mathfrak{Top}$, the category of topological spaces and continuous map between them. The second is $\mathfrak{Met}$, the category of metric spaces and short maps between them. Those results are to be read as follows. Every lax bipermutative functor F : **Isometry** $\to \mathcal{D}$ lifts uniquely to a strict bipermutative functor $\widehat{F}$: **CPTP** $\to \mathcal{D}$. Whenever $\mathcal{D}$ is enriched over $\mathfrak{Top}$ and F continuous, $\widehat{F}$ is also continuous. Whenever $\mathcal{D}$ is enriched over $\mathfrak{Met}$ and F short, $\widehat{F}$ is also short.

5.1 Enrichment over topological spaces

The homset $[n,n]$ in **Unitary** has a metric inherited by the spetral norm $\|-\|_2$, hence a topology inherited by this metric. Similarly in **Isometry**. Morphisms f in **CPTP**, **CPU**, **Cstar** have a norm given by the operator norm $\|f\|_{op} := sup_{\|a\|_2=1}\|f(a)\|_2$. Thus it naturally equips the homsets with a metric and hence a topology.

We consider the monoidal category $(\mathfrak{Top}, \times)$ of topological spaces and continuous maps with the Cartesian product as tensor.

Lemma 5.1. *With the topology given above,* **Unitary**, **Isometry**, **CPTP**, **CPU**, **Cstar** *are* $\mathfrak{Top}$*-enriched.*

Notation 5.2. *We denote by* $\mathfrak{Top}$-$\mathcal{C}$ *any of the previous categories* $\mathcal{C}$ *seen as a* $\mathfrak{Top}$*-enriched category.*

The main lemma we use is proved in (Eliott [6], lemma 3.7) and it requires the following from (Glimm [26]:

Lemma 5.3. *(Glimm [26], lemma 1.8) If $\{E_i : 1 \leq i \leq n\}$ and $\{F_i : 1 \leq i \leq n\}$ are each orthogonal families of projection in a $\mathbb{C}^*$-algebra A, and if $\|E_i - F_i\| < 1$, then there is a partial isometry W in A such that $E_i W F_i$ is a partial isometry from F_i to E_i, and W is a partial isometry from $\sum_i F_i$ to $\sum_i E_i$.*

Lemma 5.4. *(Eliott) Let $\phi, \psi : A \to B$ be unital $*$-homomorphisms between finite dimensional $\mathbb{C}^*$-algebras such that $\|\phi - \psi\| < 1$. Then there is a unitary U on B such that $\phi = ad_U \circ \psi$.*

Definition 5.5. *Given $n, \bar{m} := (m_1, \ldots, m_k)$, an $(n, \bar{m})$-Bratteli tuple $(s_1, \ldots, s_k)$ is a tuple of natural numbers such that $\sum_i s_i m_i = n$. The number of $(n, \bar{m})$-Bratteli tuples is denoted $C_n^{\bar{m}}$.*

Definition 5.6. *Given a $(n, \bar{m})$-Bratteli tuple $s_1, \ldots, s_k$, let $H_n^{\bar{m}}(s_1, \ldots, s_k) := \{ad_U \circ \varphi \circ \bigoplus_i \Delta_{s_i, m_i} : U\ n \times n \text{ unitary}\} \subset [\bigoplus_i \mathcal{M}_{m_i}, \mathcal{M}_n]$.*

Definition 5.7. *Given a lax bipermutative functor $(F, \psi) : \mathbf{Unitary} \to \mathcal{C}$ and a $(n, \bar{m})$-Bratteli tuple $\bar{s} := (s_1, \ldots, s_k)$, there is a canonical map that sends $F(U)$ to $F(U) \circ \psi \circ \bigoplus_i \Delta_{s_i, m_i}$ given by precomposition, which we denote by $can_{\bar{s}, \bar{m}, n} : [Fn, Fn] \to [\bigoplus_i Fm_i, Fn]$.*

Definition 5.8. *Let $\mathbb{U}(n)$ be the group of $n \times n$ untiary matrices. Given a $(n, \bar{m})$-Bratteli tuple $\bar{s} := (s_1, \ldots, s_k)$, let*

$$G_{\bar{s}, \bar{m}, n} := \{u_1 \otimes Id_{s_1} \oplus \ldots u_k \otimes Id_{s_k} : u_i \in \mathbb{U}(m_i)\}$$

be a subgroup of $\mathbb{U}(n)$.

$G_{\bar{s}, \bar{m}, n}$ acts on $\mathbb{U}(n)$ by right multiplication. We consider the quotient $\mathbb{U}(n)/G_{\bar{s}, \bar{m}, n}$. The following lemma is the key point to show that the lifted bipermutative morphism – written $\widehat{F}$ in the previous section – is continuous whenever the lax one (F, ψ) is:

Lemma 5.9.

- *$can_{\bar{s}, \bar{m}, n} \circ F_{n,n} : [n, n] \to [\bigoplus_i Fn_i, Fn]$ respects the quotient by $G_{\bar{s}, \bar{m}, n}$.*
- *The following are homeomorphic:*

$$\Big[\bigoplus_i \mathcal{M}_{m_i}, \mathcal{M}_n\Big] \cong \coprod_{\bar{s}:\ (n, \bar{m})\text{-Bratteli tuple}} \mathbb{U}(n)/G_{\bar{s}, \bar{m}, n}$$

Theorem 5.1. *$\mathfrak{Top} - \mathbf{Cstar}$ is the completion of $\mathfrak{Top} - \mathbf{Unitary}$ in the $\mathfrak{Top}$-enriched setting.*

$$\begin{array}{ccc} \mathfrak{Top} - \mathbf{Unitary} & \xrightarrow{(\mathbb{E}, \varphi)} & \mathfrak{Top} - \mathbf{Cstar} \\ & \searrow^{\forall (F, \psi)} & \downarrow \exists ! \widehat{F} \\ & & \forall \mathcal{C} \end{array}$$

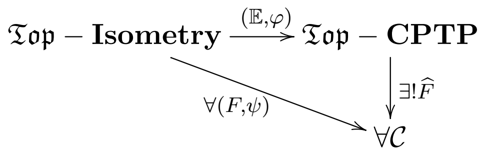
where (F, ψ) is a $\mathfrak{Top}$-enriched lax-bipermutative functor and $\widehat{F}$ a $\mathfrak{Top}$-enriched strict bipermutative functor.

Proof notes. The uniqueness is obvious from the previous section as uniqueness already holds for the Set version. The existence of a set function is ensured by the previous theorem. It remains to show it is continuous. $\widehat{F}_{n,n}$ is shown to be continuous by the lemma 5.9 and by the universal property of the quotient topology. We then construct all morphisms $\widehat{F}$ from these, the canonical maps and the universal property of the product. The resulting morphisms are all continuous by construction. □

There is no maximal Stinespring dilation for a CPTP map, but it is always possible to chose a dilation space which is common to all morphisms in $[\bigoplus_i \mathcal{M}_{m_i}, \mathcal{M}_n]$:

Lemma 5.10. *Any morphism $f \in [\bigoplus_i \mathcal{M}_{m_i}, \mathcal{M}_n]$ has a dilation in $\mathcal{M}_D$, where $D := \sum_i m_i^2 n$.*

Theorem 5.2. $\mathfrak{Top} - \mathbf{CPTP}$ *is the completion of* $\mathfrak{Top} - \mathbf{Isometry}$ *in the* $\mathfrak{Top}$*-enriched setting.*



where (F, ψ) is a $\mathfrak{Top}$-enriched lax-bipermutative functor and $\widehat{F}$ a $\mathfrak{Top}$-enriched strict bipermutative functor.

Proof notes. We prove the dual result. We proceed similarly to the proof of the previous theorem. The base case is obtained by the universal property of the quotient of the topological space of dilations $[D, n] \to [\bigoplus_i \mathcal{M}_{m_i}, \mathcal{M}_n]$ where $D = \sum_i m_i^2 n$. □

5.2 Enrichment over metric spaces

Let $\mathfrak{Met}$ be the category of metric spaces and short maps between them. There are several possible monoidal structure on $\mathfrak{Met}$. If we equip $\mathfrak{Met}$ with a tensor product $\otimes$ such that $d_{X\otimes Y} = d_X \times d_Y$ or $d_{X\otimes Y} = max(d_X, d_Y)$, then **Unitary** is not naturally enriched in $(\mathfrak{Met}, \otimes)$ as the composition fails to be short.

Definition 5.11. *If (X, d_X), (Y, d_Y) are metric spaces, we define $(X \otimes Y, d_{X\otimes Y}$ the metric space whose carrier is $X \otimes Y := X \times Y$ and whose metric is given by $d_{X\otimes Y} = d_X + d_Y$.*
Then $(\mathfrak{Met}, \otimes)$ is a monoidal category.

Lemma 5.12. **Unitary**, **Isometry**, **CPTP**, **CPU**, **Cstar** *are $(\mathfrak{Met}, \otimes)$-enriched via the metric inherited by the spetrcal and operator norms respectively.*

Notation 5.13. *We denote by $\mathfrak{Met}$-$\mathcal{C}$ any of the previous categories $\mathcal{C}$ seen as a $\mathfrak{Met}$-enriched category.*

The next issue is that the functor E : **Unitary** $\to$ **Cstar** is not enriched if we consider the metric $d(f, g) := \|f - g\|_{op}$ in **Cstar**:

Proposition 5.14. *Let $n \geq 2$. Then the map ad :* **Unitary**$(n, n) \to$ **Cstar**$(\mathcal{M}_n, \mathcal{M}_n)$ *is 2-lipschitz,* i.e. *for all $U, V \in \mathbb{U}(n)$ we have $\|ad_U - ad_V\|_{op} \leq 2\|U - V\|_2$. Furthermore, the inequality is optimal in the sense that there is no constant $c < 2$ such that ad is c-lipschitz.*

This justifies that we consider the metric $d(f, g) := \frac{1}{2}\|f - g\|_{op}$ on **Cstar**. **Cstar** is still enriched in $\mathfrak{Met}$ after this.

With the notation from the previous subsection, we can show:

Proposition 5.15. *For any $(n, \bar{m})$-Bratteli tuple $\bar{s}$, the topological space $\mathbb{U}(n)/G_{\bar{s},\bar{m},n}$ is a manifold and inherits a canonical quotient metric d'.*

Lemma 5.16. *$([\mathcal{M}_n(\mathbb{C}), [\mathcal{M}_n(\mathbb{C})], d)$ is isomorphic as a metric space to $(\mathbb{U}/\mathbb{U}(1), d')$.*

With all those ingredients we can finally show:

Theorem 5.3. $\mathfrak{Met} - \mathbf{CPTP}$ *is the completion of* $\mathfrak{Met} - \mathbf{Isometry}$ *in the* $\mathfrak{Met}$ *enriched setting.*

$$\begin{array}{ccc} \mathfrak{Met} - \mathbf{Isometry} & \xrightarrow{(\mathbb{E},\varphi)} & \mathfrak{Met} - \mathbf{CPTP} \\ & \underset{\forall(F,\psi)}{\searrow} & \downarrow \exists!\widehat{F} \\ & & \forall\mathcal{C} \end{array}$$

where (F, ψ) *is a* $\mathfrak{Met}$*-enriched lax-bipermutative functor and* $\widehat{F}$ *a* $\mathfrak{Met}$*-enriched strict bipermutative functor.*

Proof notes. Uniqueness and existence of a set function is the same as for the topological case. The key point is that the set map $\widehat{F}_{n,n}$ is proven to be short by the previous lemma and proposition. Then we can show in a sililar manner to the topological case that all the functions $\widehat{F}$ are short. □

6 Related work

Comparison with the CPM construction

The CPM construction [52] from categorical quantum mechanics [18] has been described as by a universal property as an initial object [23], which is close in spirit to our characterisation. Still, it is unclear how this initiality relates abstractly to our universal properties. The two notions of equivalence of dilation coincide when dilations are essentially unique but this is unlikely to be the case in general. Indeed it's often taken as a postulate for reconstructing quantum mechanics (e.g. [14]).

Category theory for Quantum foundations

There is a now long tradition of using category theory as a tool for investigating foundations of quantum theories [23, 21, 24], using presheaf categories as in [51, 50, 58, 41] and for the study of non-locality and contextuality abramsky2011sheaf. Our work here is more focused around Stinespring theorem in finite dimension and is in a way closer to [64] in infinite dimension where Stinespring theorem is found to satisfy a universal property.

On the other hand, our work on metric space provides a justification for the metric on superoperators and has to do with the theorietical investigation of approximation and fault tolerence, in the vein of diagramatic calculus such as ZX [7, 32, 45].

Quantum programming languages

Our work is part of a research program [29, 57] to consider quantum algebraiclly as a resource for programming languages. Several quantum programming languages have emerged [53, 3]. We would like to go even further by combining the theory of algebraic effects with quantum.

Understanding probabilistic theories

There has been a lot of recent work on understanding probabilities in and with algebraic theories [43, 42, 59]. In particular, relating probabilities from quantum and usual probability theory has attracted attention [31, 25] as well as some reconstruction of quantum theory in generalized probabilistic theories [14, 15]. Our purely categorical account will hopefully lead to some insight in these directions.

Using continuity in QM

Our theorems in the topological and metric settings are inspired by a theorem of continuity for Stinespring thereom [37]. A key notion there was the notion of fidelity[9, 38]. We wanted a more abstract point of view to understand Stinespring theorem in the context of quantum information theory.

Overall summary

We have presented a framework to study models of quantum computation (§2) and used to it prove that the category of $*$-homomorphisms is a completion of the category of unitaries (§3). We showed that the category of quantum channels is the same completion but of the category of isometries, which makes Von Neumann's model canonical (§4), and extended these results to two enriched settings (§5).

7 Conclusion and further work

As was shown in the previous section, there are several related work and it could be fruitful to compare them with our work in more depth. In particular, a version of our theorem in infinite dimension using Paschke dilations [64] would be great. More generally the importance of Von Neuman algebras as opposed to general $\mathbb{C}^*$-algebras could be highlighted in a variant of our topologically enriched framework. I would to explore some of these questions and some other during my PhD. In particular, such a general framework could be great to explore monads for quantum, be it local state for memory management or more related to probabilities with variants of the Giry or the expectation monads.

I was awarded a full scholarship at Oxford University to pursue a PhD there. I accepted this scholarship and will continue to work under the supervision of Sam Staton – starting in October – on using categorical tools for studying quantum mechanics and its applications to quantum information and quantum computing.

Acknowledgement

I would like to thank Bas Westerbaan, Paul-Blain Levy, Dario Stein, Sam Speight, Younesse Kaddar and Ohad Kammar for useful discussions. I am very grateful to my supervisor Sam Staton.

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Appendix

Definitions

Definition .1. *A bipermutative category comprises a category* $\mathcal{C}$ *with two symmetric strict monoidal structures,* $(\oplus, \gamma, N)$ *and* $(\otimes, \gamma', I)$ *together with natural isomorphisms* $\lambda^* : N \otimes A \cong N$, $\rho^* : A \otimes N \cong N$, $\delta : (A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$, $\delta^\sharp : A \otimes (B \oplus C) \cong (A \otimes B) \oplus (A \otimes C)$*, such that* λ^*, ρ^* *and* δ *are identity morphisms between equal objects, and the following diagrams commute:*

$$
\begin{array}{ccc}
A \otimes (B \oplus C) & \xrightarrow{A \otimes \gamma} & A \otimes (C \oplus B) \\
\downarrow{\scriptstyle \delta^\sharp} & & \downarrow{\scriptstyle \delta^\sharp} \\
(A \otimes B) \oplus (A \otimes C) & \xrightarrow[\gamma]{} & (A \otimes C) \oplus (A \otimes B)
\end{array}
$$

$$
\begin{array}{ccc}
A \otimes (B \oplus C) & \xrightarrow{\gamma'} & (B \oplus C) \otimes A \\
\downarrow{\scriptstyle \delta^\sharp} & & \downarrow{\scriptstyle \delta} \\
(A \otimes B) \oplus (A \otimes C) & \xrightarrow[\gamma' \oplus \gamma']{} & (B \otimes A) \oplus (C \otimes A)
\end{array}
$$

$$
\begin{array}{ccc}
(A \oplus B) \otimes (C \oplus D) & \xrightarrow{\delta^\sharp} & ((A \oplus B) \otimes C) \oplus ((A \oplus B) \otimes D) \\
\downarrow{\scriptstyle \delta} & & \\
(A \otimes (C \oplus D)) \oplus (B \otimes (C \oplus D)) & & \downarrow{\scriptstyle \delta \oplus \delta} \\
\downarrow{\scriptstyle \delta^\sharp \oplus \delta^\sharp} & & \\
(A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (B \otimes D) & \xrightarrow[(A \otimes C) \oplus \gamma \oplus (B \otimes D)]{} & (A \otimes C) \oplus (B \otimes C) \oplus (A \otimes D) \oplus (B \otimes D)
\end{array}
$$

Definition .2. *A* $\oplus$-lax *bipermutative functor is a functor* $F : \mathcal{A} \to \mathcal{B}$ *between bipermutative categories such that* $F(N) = N$, $F(I) = I$, $F(A \otimes B) = F(A) \otimes F(B)$, $F(\gamma') = \gamma'$ *together with a natural transformation* $\varphi : F(A) \oplus F(B) \to F(A \oplus B)$ *such that the following diagrams commute:*

$$
\begin{array}{ccc}
(F(A) \oplus F(B)) \otimes F(C) & = & (F(A) \otimes F(C)) \oplus (F(B) \otimes F(C)) \\
\downarrow{\scriptstyle \varphi \otimes F(C)} & & \downarrow{\scriptstyle \varphi} \\
F(A \oplus B) \otimes F(C) & = & F((A \otimes C) \oplus (B \otimes C))
\end{array}
$$

$$
\begin{array}{ccc}
F(A) \oplus F(B) & \xrightarrow{\varphi_{A,B}} & F(A \oplus B) \\
\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle F(\gamma)} \\
F(B) \oplus F(A) & \xrightarrow[\varphi_{B,A}]{} & F(B \oplus A)
\end{array}
$$

$$
\begin{array}{ccc}
F(A) \oplus F(B) \oplus F(C) & \xrightarrow{F(A) \oplus \varphi} & F(A) \oplus F(B \oplus C) \\
\downarrow{\scriptstyle \varphi \oplus F(C)} & & \downarrow{\scriptstyle \varphi} \\
F(A \oplus B) \oplus F(C) & \xrightarrow[\varphi]{} & F(A \oplus B \oplus C)
\end{array}
$$

$$
\begin{array}{ccc}
F(A) \oplus F(N) & \xrightarrow{\varphi_{A,N}} & F(A \oplus N) \\
\| & & \| \\
F(A) & = & F(A)
\end{array}
$$