# Quantum channels as a categorical completion

ENS Paris-Saclay

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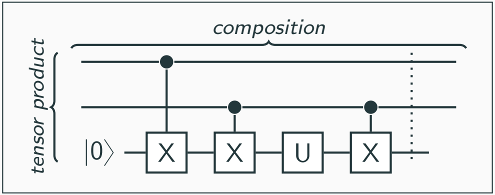
# Introduction

# Standard point of view

- pure QM is not random, and is reversible
- pure QM does not allow discarding



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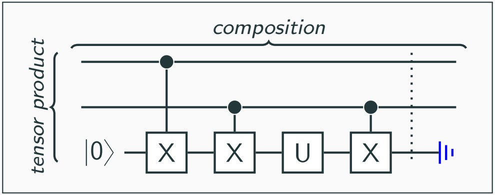
- pure QM does not allow discarding
- full QM: mixed states, quantum channels



# Von Neumann's model: density matrices

Pure QM

- state space $\mathbb{C}^n$
- combination of systems: $\otimes$
- ancilla (auxiliary system)
- unitary transformation $U$

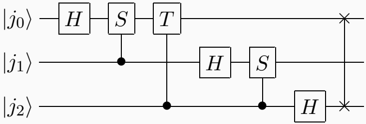



**Figure 1:** Quantum Fourier transform on three qubits

Completely Positive Trace Preserving (CPTP) maps

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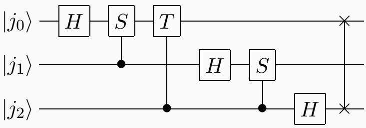


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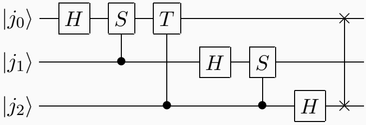


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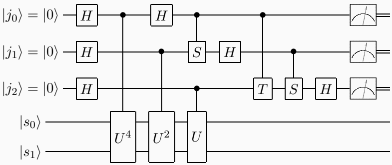


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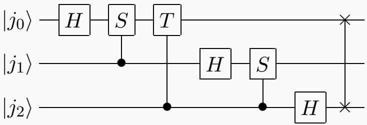


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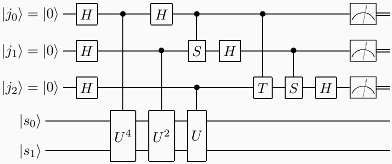


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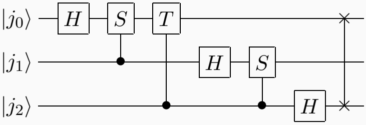


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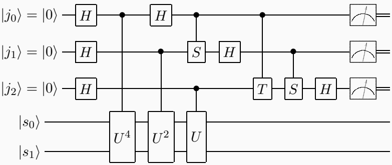


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# Today's presentation

**Informally:**
The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM, quotients global phase and allows discarding.

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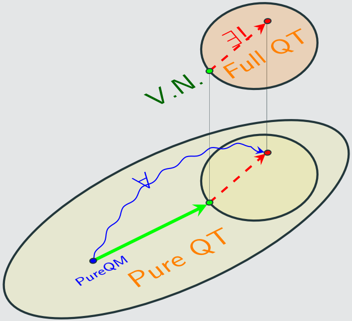
The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM, quotients global phase and allows discarding.

**CPTP is the universal monoidal category on PureQM whose unit is a terminal object:**



# Outline

Introduction

Symmetric monoidal categories with discarding

Universality of CPTP

Interpretation and summary

# Category

A category $\mathcal{C}$ is given by a class of objects $Obj(\mathcal{C})$ and a class of arrows $Morph(\mathcal{C})$ such that:

- each arrow $f$ has a domain and a codomain object: $f : A \to B$
- for each object $A$, there is a morphism $id_A : A \to A$
- for each $f : A \to B, g : B \to C$, there is a morphism $gf : A \to C$
- the composition is associative and $id$ is a neutral

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**Example: sets and functions**

**Example: real vector spaces and linear maps**

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**Example: sets and functions**

**Example: real vector spaces and linear maps**

**Examples: PureQM and Quantum channels**

Each object is a state space of a certain dimension and each morphism a valid transformation.

# Symmetric strict monoidal category

A symmetric (strict) monoidal category $(\mathcal{C}, \otimes, I, \sigma)$ is a catgory $\mathcal{C}$ with the following extra structure:

- for every objects $A, B$, an object $A \otimes B$
- for every morphisms $f : A \to B, g : C \to D$, a morphism $f \otimes g : A \otimes C \to B \otimes D$
- for every objects $A, B$, a natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$
- a distinguished object $I$

such that:

- $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
- $\lambda_A : I \otimes A = A = A \otimes I$
- $\sigma_{A,B} = \sigma^{-1}_{B,A}$

and certain other coherence conditions.

# The category Isometry (PureQM)

We define the category **Isometry** as follows:

- Objects: natural numbers $n$ ($\mathbb{C}^n$)
- Morphisms $f : n \to m$ are linear maps $f : \mathbb{C}^n \to \mathbb{C}^m$ that are isometries: $\forall v, ||f(v)|| = ||v||$
- Composition: composition of linear maps
- $m \otimes n := mn$ and $f \otimes g$ is the usual tensor product

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**Examples:**

- the isometries $V : n \to n$ are the unitaries
- an isometry $V : 1 \to n$ is a pure state

# Discarding

**Monoidal category with discarding:**
A (strict) symmetric monoidal category $(\mathbf{C}, \otimes, I)$ has discarding when the unit of the tensor product $I$ is a terminal object.

$$\begin{array}{ccc} A & \xrightarrow{!} & 1 \\ {\scriptstyle f}\downarrow & \nearrow_{!} & \\ B & & \end{array}$$

# The category CPTP (FullQM)

We define the category **CPTP** of completely positive trace preserving maps as follows:

- Objects are natural numbers $n$ $(\mathcal{M}_n(\mathbb{C}))$
- Morphisms $f : n \to m$ are linear maps $f : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ that are completely positive and trace preserving
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**CPTP has discarding**

- $!_n : n \to 1$ is the trace operator
- $id_m \otimes !_n : m \otimes n \to m \otimes 1 = m$ is the partial trace operator

# Symmetric strict monoidal functor

A functor is a morphism of categories. In detail a functor $F : \mathcal{C} \to \mathcal{D}$ is given by:

- for every object $A \in \mathcal{C}$, an object $FA \in \mathcal{D}$
- for every morphism $f : A \to B \in \mathcal{C}$, a morphism $Ff : FA \to FB \in \mathcal{D}$

such that $F(id_A) = id_{FA}$ and $F(f \circ g) = Ff \circ Fg$.

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Similarly, a symmetric monoidal functor $F$ is a morphism of symmetric monoidal categories. It satisfies:

- $F(A \otimes B) = F(A) \otimes F(B)$
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**$E$ : Isometry $\to$ CPTP**

- $E(n) := n$
- $E(V) := ad_V : M \mapsto VMV^*$

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# Main theorem

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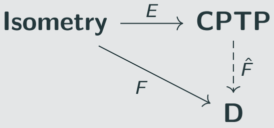
**Theorem: universality of CPTP**

- $\forall$ **D** strict symmetric monoidal category with discarding
- $\forall$ $F$ : **Isometry** $\to$ **D** symmetric strict monoidal functor

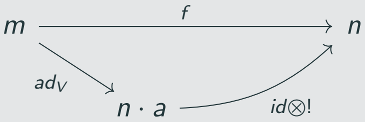
There is a unique symmetric monoidal functor $\hat{F}$ : **CPTP** $\to$ **D** such that:



# Proof: key lemma

**Stinespring theorem**

For every CPTP $f$ there is a pair $(V, a)$ such that:



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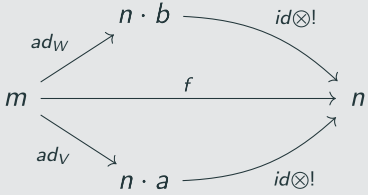

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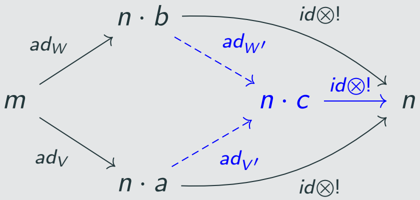

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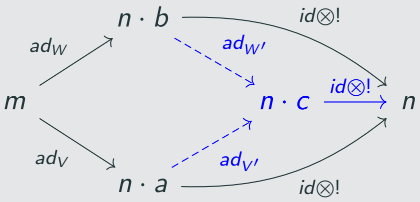


$(V, a)$ is called a dilation for $f$.

# Proof: uniqueness

If any symmetric monoidal functor $\hat{F}$ is going to make diagram commute then it must be defined as

- $\hat{F}(n) \stackrel{\text{def}}{=} F(n)$ as $E$ is identity on objects
- If $$\begin{array}{ccc} n & \xrightarrow{\quad f \quad} & m \\ & {\scriptstyle ad_V}\searrow \qquad \nearrow{\scriptstyle id\otimes !} & \\ & n \cdot a & \end{array}$$ then $$\begin{array}{ccc} n & \xrightarrow{\quad \hat{F}(f) \quad} & m \\ & {\scriptstyle \hat{F}(ad_V)}\searrow \qquad \nearrow{\scriptstyle \hat{F}(id\otimes !)} & \\ & n \cdot a & \end{array}$$

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# Proof: well-definedness

Only choice :

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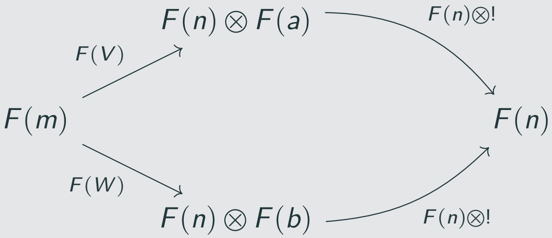
**Independence of the choice of dilation** $(V, a)$

Given $(W, b)$ another dilation,



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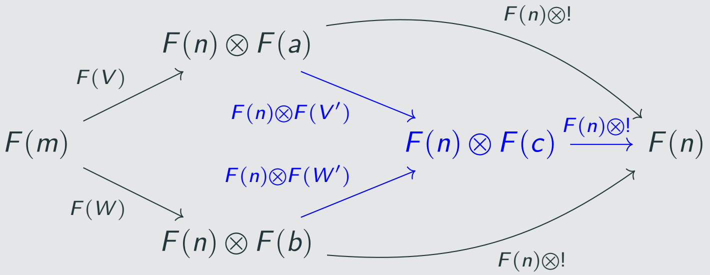
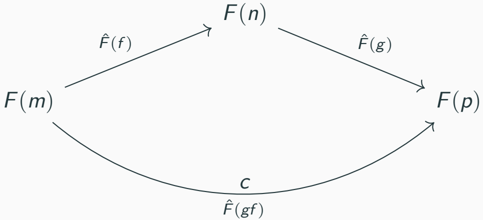
Given $(W, b)$ another dilation,Stinespring theorem guarantees there is a triple $(c, V', W')$ such that:



# Proof: functoriality

- Identity: dilation $(id_n, 1)$
- Composition: if $(V, a)$ is a dilation of $f : m \to n$ and $(W, b)$ is a dilation of $g : n \to p$,



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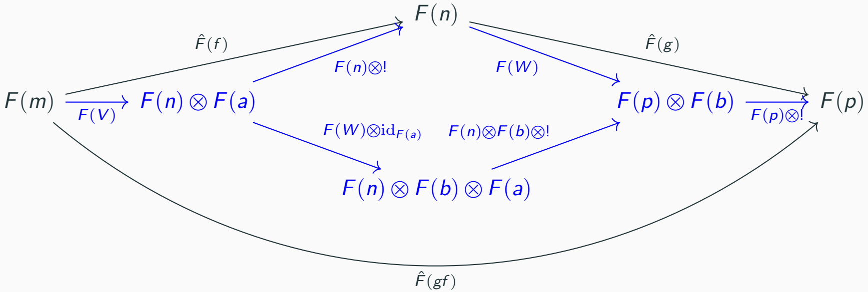

# Proof: monoidal functor

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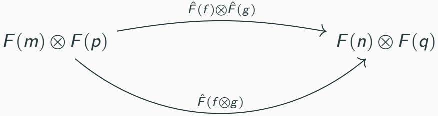

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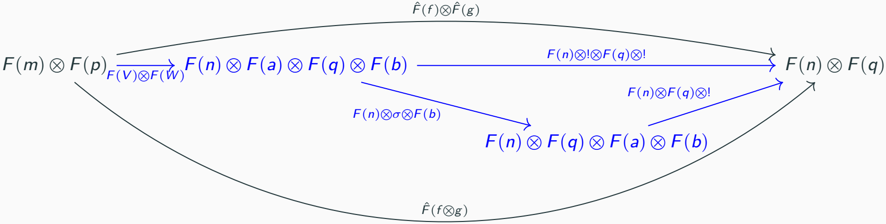

# Outline

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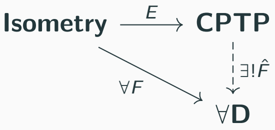
Universality of CPTP

Interpretation and summary

# Interpretation of the universality

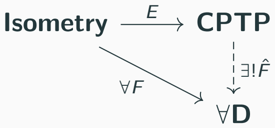
- foundational justification for the model
- new definition for CPTP
- relies on Stinespring theorem (purification uniqueness)



# Summary and other work

**Summary:**

- CPTP are canonical (universal property)
- Motivated by physics arguments



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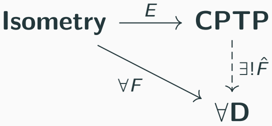


**Not presented:**

- bipermutative categories
- categories enriched over topological spaces and metric spaces
- syntactical completeness of Staton's theory
- link to affine reflections