Quantum channels as a categorical completion

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Introduction
• pure QM is not random, and is reversible
• pure QM does not allow discarding
Standard point of view

- pure QM is not random, and is reversible
- pure QM does not allow discarding
- full QM: mixed states, quantum channels
**Von Neumann’s model: density matrices**

**Pure QM**
- state space $\mathbb{C}^n$
- combination of systems: $\otimes$
- ancilla (auxiliary system)
- unitary transformation $U$

**Completely Positive Trace Preserving (CPTP) maps**
- $\mathcal{M}_n(\mathbb{C})$
- combination of systems: $\otimes$
- ancilla
- $ad_U : M \leftrightarrow UMU^*$ super-operator

**Figure 1:** Quantum Fourier transform on three qubits
Von Neumann’s model: density matrices

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- allows discarding (trace)

**Figure 1:** Quantum Fourier transform on three qubits

**Figure 2:** Three-qubit phase estimation circuit with QFT and controlled-U
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- state space $\mathbb{C}^n$
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$|j_0\rangle$ \hspace{1cm} $|j_1\rangle$ \hspace{1cm} $|j_2\rangle$

$H$ \hspace{1cm} $S$ \hspace{1cm} $T$

$H$ \hspace{1cm} $S$

$H$

Figure 1: Quantum Fourier transform on three qubits

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$|j_0\rangle = |0\rangle$

$|j_1\rangle = |0\rangle$

$|j_2\rangle = |0\rangle$

$|s_0\rangle$

$U^4$

$U^2$

$U$

Figure 2: Three-qubit phase estimation circuit with QFT and controlled-U
Von Neumann’s model: density matrices

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\[
\begin{align*}
|j_0\rangle &= HST |j_0\rangle \\
|j_1\rangle &= HS |j_1\rangle \\
|j_2\rangle &= H |j_2\rangle
\end{align*}
\]

Figure 1: Quantum Fourier transform on three qubits

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**Completely Positive Trace Preserving (CPTP) maps**
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**Figure 2:** Three-qubit phase estimation circuit with QFT and controlled-U
Informally:
The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM, quotients global phase and allows discarding.
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The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM, quotients global phase and allows discarding.

CPTP is the universal monoidal category on PureQM whose unit is a terminal object:
Outline

Introduction

Symmetric monoidal categories with discarding

Universality of CPTP

Interpretation and summary
A category $\mathcal{C}$ is given by a class of objects $\text{Obj}(\mathcal{C})$ and a class of arrows $\text{Morph}(\mathcal{C})$ such that:

- each arrow $f$ has a domain and a codomain object: $f : A \to B$
- for each object $A$, there is a morphism $\text{id}_A : A \to A$
- for each $f : A \to B$, $g : B \to C$, there is a morphism $gf : A \to C$
- the composition is associative and $\text{id}$ is a neutral
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**Example: sets and functions**
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**Example:** sets and functions

**Example:** real vector spaces and linear maps
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Example: sets and functions

Example: real vector spaces and linear maps

Examples: PureQM and Quantum channels

Each object is a state space of a certain dimension and each morphism a valid transformation.
A symmetric (strict) monoidal category \((\mathcal{C}, \otimes, I, \sigma)\) is a category \(\mathcal{C}\) with the following extra structure:

- for every objects \(A, B\), an object \(A \otimes B\)
- for every morphisms \(f : A \to B, g : C \to D\), a morphism \(f \otimes g : A \otimes C \to B \otimes D\)
- for every objects \(A, B\), a natural isomorphism \(\sigma_{A,B} : A \otimes B \to B \otimes A\)
- a distinguished object \(I\)

such that:

- \((A \otimes B) \otimes C = A \otimes (B \otimes C)\)
- \(\lambda_A : I \otimes A = A = A \otimes I\)
- \(\sigma_{A,B} = \sigma_{B,A}^{-1}\)

and certain other coherence conditions.
The category Isometry (PureQM)

We define the category **Isometry** as follows:

- **Objects**: natural numbers \( n \) \((\mathbb{C}^n)\)
- **Morphisms** \( f : n \to m \) are linear maps \( f : \mathbb{C}^n \to \mathbb{C}^m \) that are isometries: \( \forall v, \| f(v) \| = \| v \| \)
- **Composition**: composition of linear maps
- \( m \otimes n := mn \) and \( f \otimes g \) is the usual tensor product
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Equivalently morphisms are matrices $V$ such that $V^*V = I$ and $\otimes$ is then the Kronecker product.
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**Examples:**
- the isometries $V : n \rightarrow n$ are the unitaries
- an isometry $V : 1 \rightarrow n$ is a pure state
Monoidal category with discarding:
A (strict) symmetric monoidal category \((\mathbf{C}, \otimes, I)\) has discarding when the unit of the tensor product \(I\) is a terminal object.

\[
\begin{array}{ccc}
A & \xrightarrow{!} & 1 \\
\downarrow f & & \\
B & \xrightarrow{!} & \\
\end{array}
\]
We define the category **CPTP** of completely positive trace preserving maps as follows:

- **Objects** are natural numbers \( n \) \((\mathcal{M}_n(\mathbb{C}))\)
- **Morphisms** \( f : n \rightarrow m \) are linear maps \( f : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_m(\mathbb{C}) \) that are completely positive and trace preserving
- **Composition**: composition of linear maps
- **\( m \otimes n := mn \)** and \( f \otimes g \) is again the tensor product
The category CPTP (FullQRM)

We define the category **CPTP** of completely positive trace preserving maps as follows:

- **Objects** are natural numbers $n (\mathcal{M}_n(\mathbb{C}))$
- **Morphisms** $f : n \rightarrow m$ are linear maps $f : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_m(\mathbb{C})$ that are completely positive and trace preserving
- **Composition**: composition of linear maps
- $m \otimes n := mn$ and $f \otimes g$ is again the tensor product

### CPTP has discarding

- $!_n : n \rightarrow 1$ is the trace operator
- $id_m \otimes !_n : m \otimes n \rightarrow m \otimes 1 = m$ is the partial trace operator
A functor is a morphism of categories. In detail a functor $F : C \to D$ is given by:

- for every object $A \in C$, an object $FA \in D$
- for every morphism $f : A \to B \in C$, a morphism $Ff : FA \to FB \in D$

such that $F(id_A) = id_{FA}$ and $F(f \circ g) = Ff \circ Fg$. 

Similarly, a symmetric monoidal functor $F$ is a morphism of symmetric monoidal categories. It satisfies:

- $F(p_A b B) = F(p_A) q_B F(p_B q)$
- $F(p_I q) = I$ with certain coherence conditions.

$\text{E}$: Isometry $\to \text{CPTP}$

$E(p_n q) : n$ $E(p_V q) : ad V : M \Rightarrow VMV$
A functor is a morphism of categories. In detail a functor $F : C \to D$ is given by:

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Similarly, a symmetric monoidal functor $F$ is a morphism of symmetric monoidal categories. It satisfies:

- $F(A \otimes B) = F(A) \otimes F(B)$
- $F(I) = I$

with certain coherence conditions.
A functor is a morphism of categories. In detail a functor $F : C \to D$ is given by:

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Similarly, a symmetric monoidal functor $F$ is a morphism of symmetric monoidal categories. It satisfies:

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- $F(I) = I$

with certain coherence conditions.

$E : \text{Isometry} \to \text{CPTP}$

- $E(n) := n$
- $E(V) := ad_V : M \leftrightarrow VMV^*$
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Symmetric monoidal categories with discarding

Universality of CPTP

Interpretation and summary
Main theorem

The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM, quotients global phase and allows discarding.
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**Theorem: universality of CPTP**

- \( \forall D \) strict symmetric monoidal category with discarding
- \( \forall F : \text{Isometry} \rightarrow D \) symmetric strict monoidal functor

There is a unique symmetric monoidal functor \( \hat{F} : \text{CPTP} \rightarrow D \) such that:

\[
\begin{array}{ccc}
\text{Isometry} & \xrightarrow{E} & \text{CPTP} \\
\downarrow F & & \downarrow \hat{F} \\
D & & D
\end{array}
\]
Proof: key lemma

**Stinespring theorem**

For every CPTP $f$ there is a pair $(V, a)$ such that:

$$m \xrightarrow{id} \underbrace{aV}_{\text{ad}_V} \xrightarrow{n \cdot a} \xrightarrow{id} n$$
Proof: key lemma

Stinespring theorem

For every CPTP $f$ there is a pair $(V, a)$ such that:

\[
\begin{array}{ccc}
  m & \xrightarrow{ad_W} & n \cdot b \\
  \downarrow f & & \downarrow id \otimes ! \\
  n \cdot a & \xleftarrow{ad_V} & n
\end{array}
\]
Stinespring theorem

For every CPTP $f$ there is a pair $(V, a)$ such that:
Proof: key lemma

**Stinespring theorem**

For every CPTP $f$ there is a pair $(V, a)$ such that:

$$(V, a) \text{ is called a dilation for } f.$$
Proof: uniqueness

If any symmetric monoidal functor \( \hat{F} \) is going to make diagram commute then it must be defined as

- \( \hat{F}(n) \overset{\text{def}}{=} F(n) \) as \( E \) is identity on objects

- If \( \quad n \xrightarrow{f} m \quad \) then \( \quad n \cdot a \xrightarrow{\hat{F}(f)} m \)

\[
\begin{array}{ccc}
  n & \xrightarrow{f} & m \\
  \downarrow^{ad_V} & \quad & \quad \downarrow^{id \otimes !} \\
  n \cdot a & \xrightarrow{id \otimes !} & m
\end{array}
\]

\[
\begin{array}{ccc}
  n & \xrightarrow{\hat{F}(f)} & m \\
  \downarrow^{\hat{F}(ad_V)} & \quad & \quad \downarrow^{\hat{F}(id \otimes !)} \\
  n \cdot a & \xrightarrow{\hat{F}(id \otimes !)} & m
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\]
Proof: uniqueness

If any symmetric monoidal functor \( \hat{F} \) is going to make diagram commute then it must be defined as

- \( \hat{F}(n) \overset{\text{def}}{=} F(n) \) as \( E \) is identity on objects

- If \( \hat{F}(f) \) then

\[
\begin{align*}
\begin{array}{ccc}
n & \overset{f}{\rightarrow} & m \\
\downarrow_{ad_V} & & \downarrow_{id \otimes !} \\
n \cdot a & \rightarrow & n \cdot a
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
n & \overset{\hat{F}(f)}{\rightarrow} & m \\
\downarrow_{F(V)} & \overset{(id \otimes !)}{\rightarrow} & \downarrow_{(id \otimes !)} \\
n \cdot a & \rightarrow & n \cdot a
\end{array}
\end{align*}
\]
Proof: well-definedness

Only choice:

• \( \hat{F}(n) \overset{\text{def}}{=} F(n) \)
• \( \hat{F}((id \otimes!) \circ ad_V) \overset{\text{def}}{=} (id \otimes!) \circ F(V) \)
Proof: well-definedness

Only choice:

- $\hat{F}(n) \overset{\text{def}}{=} F(n)$
- $\hat{F}((id \otimes !) \circ ad_V) \overset{\text{def}}{=} (id \otimes !) \circ F(V)$

Independence of the choice of dilation $(V, a)$

Given $(W, b)$ another dilation,

\[
\begin{array}{c}
F(m) \\
F(n) \otimes F(a) \\
F(V) \\
F(W)
\end{array}
\quad \xrightarrow{F(n) \otimes !} \quad
\begin{array}{c}
F(n) \otimes F(b) \\
F(n) \otimes !
\end{array}
\]
Proof: well-definedness

Only choice:

- \( \hat{F}(n) \xrightarrow{\text{def}} F(n) \)
- \( \hat{F}((id \otimes!) \circ ad_V) \xrightarrow{\text{def}} (id \otimes!) \circ F(V) \)

**Independence of the choice of dilation \((V, a)\)**

Given \((W, b)\) another dilation, Stinespring theorem guarantees there is a triple \((c, V', W')\) such that:
Proof: functoriality

- Identity: dilation \((id_n, 1)\)
- Composition: if \((V, a)\) is a dilation of \(f : m \rightarrow n\) and \((W, b)\) is a dilation of \(g : n \rightarrow p\),
Proof: functoriality

- Identity: dilation \((id_n, 1)\)
- Composition: if \((V, a)\) is a dilation of \(f : m \to n\) and \((W, b)\) is a dilation of \(g : n \to p\), then \(((W \otimes id_a) V, b \otimes a)\) is a dilation of \(gf\).
Proof: monoidal functor

- \((V, a)\) is a dilation of \(f : m \to n\)
- \((W, b)\) is a dilation of \(g : p \to q\)

Then \(((\text{id}_m \otimes \sigma \otimes \text{id}_p) \circ (V \otimes W), a \otimes b)\) is a dilation of \(f \otimes g\).
Proof: monoidal functor

- \((V, a)\) is a dilation of \(f : m \to n\)
- \((W, b)\) is a dilation of \(g : p \to q\)

Then \(((\text{id}_m \otimes \sigma \otimes \text{id}_p) \circ (V \otimes W), a \otimes b)\) is a dilation of \(f \otimes g\).
Outline

Introduction

Symmetric monoidal categories with discarding

Universality of CPTP

Interpretation and summary
• foundational justification for the model
• new definition for CPTP
• relies on Stinespring theorem (purification uniqueness)
Summary:

- CPTP are canonical (universal property)
- Motivated by physics arguments

\begin{align*}
\text{Isometry} & \quad E \quad \text{CPTP} \\
\forall F & \quad \exists ! \hat{F} \\
\forall \mathcal{D} & 
\end{align*}
Summary and other work

Summary:
- CPTP are canonical (universal property)
- Motivated by physics arguments

Not presented:
- bipermutative categories
- categories enriched over topological spaces and metric spaces
- syntactical completeness of Staton’s theory
- link to affine reflections