Quantum Channels as a Categorical Completion

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Oxford University

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Introduction

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Some goals of quantum computing

- Quantum as a resource for computation
- Combine quantum mechanics and classical CS theory

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Are quantum channels a good framework?

Main result

A new link between pure and full quantum worlds:

Informally:
The category of Quantum Channels is the simplest (universal) category that interprets PureQM with preparations and admits discarding.

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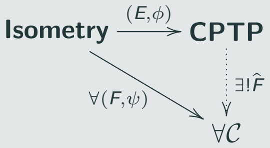
Theorem: Universality of Quantum Channels

- $\forall\ \mathcal{C}$ bipermutative category with discarding
- $\forall\ F : \mathbf{Isometry} \to \mathcal{C}$ $\oplus$-colax bipermutative functor

There is a unique strict bipermutative functor $\widehat{F} : \mathbf{CPTP} \to \mathcal{C}$ such that the following diagram commutes:



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Quantum Computing

Pure quantum computation

Reversible computation:

Classical

- $n \in \mathbb{N}$ seen as finite sets
- $n \to n$ bijections
- $n \oplus m := n + m$
- $n \otimes m := n \times m$

Quantum

- $n \in \mathbb{N}$ seen as $\mathbb{C}^n$
- $n \to n$ unitary $n \times n$ matrix
- $n \oplus m := n + m$
  allows conditional operations
- $n \otimes m := n \times m$
  allows spatial juxtaposition

Pure quantum computation

Fundamental unit of computation: qubit

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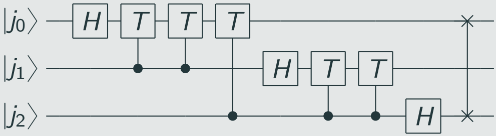

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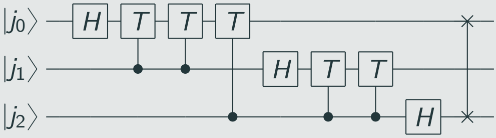

Pure QM: Unitaries $U : n \to n$

$$U^* U = UU^* = I$$

- $U \oplus V := \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$
- unit of $\oplus$: number zero 0
- $U \otimes V := \begin{bmatrix} u_{1,1}V & u_{1,2}V \\ u_{2,1}V & u_{2,2}V \end{bmatrix}$
- unit of $\otimes$: the number 1

Full quantum computing

Pure QM	Completely Positive Trace Preserving (CPTP) maps
state space $\mathbb{C}^n \ni v$	$\mathcal{M}_n(\mathbb{C}) \ni vv^T$
combination of systems: $\otimes$	combination of systems: $\otimes$
unitary transformation U	$ad_U : M \mapsto UMU^*$ super-operator

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	no global phase

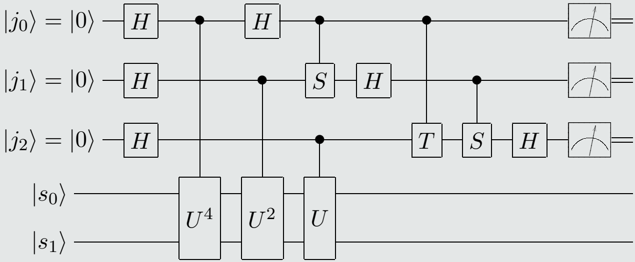
Full quantum computing

New features:

Preparating a state

Measuring a qubit

Example (phase estimation circuit):



Full quantum computing

Complex $*$-algebra: A is a complex algebra with an additional operation $*: A \to A$ such that $\mathbf{x}^{**} = \mathbf{x}$, $(a\mathbf{x})^* = \bar{a}\mathbf{x}^*$, $(\mathbf{x}+\mathbf{y})^* = \mathbf{x}^* + \mathbf{y}^*$, $(\mathbf{x}\mathbf{y})^* = \mathbf{y}^*\mathbf{x}^*$ and $1^* = 1$ for all complex numbers a and all $\mathbf{x}, \mathbf{y} \in A$, where $\bar{a}$ is the complex conjugate of a.

$\mathcal{M}_n(\mathbb{C})$ have a $*$-algebra structure given by the conjugate transpose and $\bigoplus_{j\in J} \mathcal{M}_{n_j}(\mathbb{C})$ again inherits $*$-algebra structure componentwise. Such algebras can be equipped with the spectral norm and they are complete with respect to this norm. These are finite dimensional C*-algebras and every finite dimensional C*-algebra is of this form, up to isomorphism. We will therefore from now on use 'C*-algebra' to mean a finite dimensional C*-algebra of the form $\bigoplus_{j\in J} \mathcal{M}_{n_j}(\mathbb{C})$.

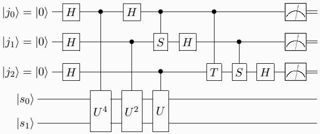
Positive element: A in a C*-algebra is such that there exists an element B such that $A = B^* B$, or equivalently if it is self-adjoint and its spectrum $\sigma(A)$ consists of non-negative real numbers.

Positive linear map: $f: A \to B$ between two C*-algebras if it maps positive elements to positive elements. , i.e. $a \geqslant 0 \Rightarrow f(a) \geqslant 0$.

Completely positive: A linear map $f: A \to B$ between two C*-algebras is *completely positive* (CP) if for every k the map $\mathrm{id}_{\mathcal{M}_k(\mathbb{C})} \otimes f: \mathcal{M}_k(\mathbb{C}) \otimes A \to \mathcal{M}_k(\mathbb{C}) \otimes B$ is positive.

Trace-presering map: Let Tr bethe trace operator. Then a linear map $f: \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ is said to be trace-preserving if for all $M \in \mathcal{M}_n(\mathbb{C})$, $\mathrm{Tr}(f(M)) = \mathrm{Tr}(M)$. We can extend the trace operator to C*-algebras by $\mathrm{Tr}(A_1, \ldots, A_k) = \mathrm{Tr}(A_1) + \ldots \mathrm{Tr}(A_k)$ which is to say we embed $\bigoplus_{i\in I} \mathcal{M}_{n_i}(\mathbb{C})$ into $\mathcal{M}_{\sum_i n_i}(\mathbb{C})$ as block diagonal matrices $((A_1, \ldots, A_k) \mapsto A_1 \oplus \ldots \oplus A_k)$ and then apply the usual trace operator.

Bipermutative categories

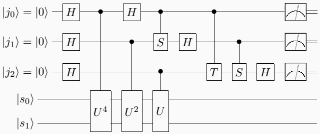


Bipermutative category:

A bipermutative category is a category with two symmetric strict monoidal structures $(\oplus, N, \gamma)$ and $(\otimes, I, \gamma')$ s.t. one distributes over the other:

- $\delta_{A,B,C} : A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$
- $N = N \otimes A$

satisfying some coherence conditions.



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Example:

- objects: natural numbers $n \in \mathbb{N}$
- morphisms: bijections $n \to n$
- $n \oplus m := n + m$
- $n \otimes m := n \times m$

Main Examples:

Bipermutative category of Unitaries: Unitary

- Objects: natural numbers $n \in \mathbb{N}$
- Morphisms: unitaries $\mathbb{C}^n \to \mathbb{C}^n$
- Composition: matrix multiplication
- $n \oplus m := n + m$ and $U \oplus V : n \oplus m \to n \oplus m$ is $\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$.

  The unit N is the number 0.
- $n \otimes m := n \times m$ and $U \otimes V : n \otimes m \to n \otimes m$ be the Kronecker product of matrices, with $(U \otimes V)_{in+k,jn+l} = U_{i,j} V_{k,l}$.
  The unit I is the number 1.

Main Examples:

Bipermutative category of Quantum channels: CPTP

- Objects:

  $[n_1, \ldots, n_k], \quad n_i > 0$

  $$[[(n_1, \ldots, n_k)]] := \bigoplus_i \mathcal{M}_{n_i}(\mathbb{C})$$

- $$f\colon [n_1, \ldots, n_k] \to [m_1, \ldots, m_p]$$

  $$f\colon \bigoplus_i \mathcal{M}_{n_i}(\mathbb{C}) \to \bigoplus_j \mathcal{M}_{m_j}(\mathbb{C}) \qquad \text{CPTP}$$

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- $[n_1, \ldots, n_k] \oplus [m_1, \ldots, m_p] := [n_1, \ldots, n_k, m_1, \ldots, m_p]$

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- $[n_1, \dots, n_k] \oplus [m_1, \dots, m_p] := [n_1, \dots, n_k, m_1, \dots, m_p]$
- $[n_1, \dots, n_k] \otimes [m_1, \dots, m_p] := [n_1 m_1, \dots, n_1 m_p, \dots, n_k m_1, \dots, n_k m_p]$
- $[[(\bar{n}) \otimes (\bar{m})]] \cong [[(\bar{n})]] \otimes [[(\bar{m})]]$

Main Examples:

Bipermutative category of Quantum channels: CPTP

The terminal object is $[1]$ $(\mathbb{C})$ and $!_A : A \to [1]$ is the trace operator.

The map $!_n \otimes \mathrm{id}_m : [nm] \to [m]$ is usually called the partial trace.

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Measurement is given by:

$$\varphi_{n,m} : \mathcal{M}_{n+m}(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{M}_m(\mathbb{C})$$
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto (A, D)$$

Bipermutative functors

Given two bipermutative categories $(\mathbf{C}, \oplus, N, \gamma, \otimes, I, \gamma')$, $(\mathbf{D}, \oplus, N, \gamma, \otimes, I, \gamma')$.

Strict bipermutative functor

$F : \mathbf{C} \to \mathbf{D}$ between bipermutative categories such that:

- $F(I) = I$,
- $F(A \otimes B) = F(A) \otimes F(B)$
- $F(\gamma'_{A,B}) = \gamma'_{FA,FB}$

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- $F(N) = N$
- $F(A \oplus B) = F(A) \oplus F(B)$
- $F(\gamma_{A,B}) = \gamma_{FA,FB}$.

Bipermutative functors

$\oplus$-colax bipermutative functor

$F : \mathbf{C} \to \mathbf{D}$ such that:

$$F(I) = I \qquad \varphi : F(A \oplus B) \to F(A) \oplus F(B)$$

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$$F(\gamma') = \gamma'$$

with certain coherence diagrams. In particular those making (F, φ, ψ) a symmetric colax monoidal functor for $(\oplus, N)$.

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$\oplus$-colax-bipermutative functor E : Isometry $\to$ CPTP

- $n \mapsto [n]$
- $\left(V : \mathbb{C}^m \to \mathbb{C}^n\right) \mapsto \left(\mathrm{Ad}_V : M \mapsto VMV^* : \mathcal{M}_m(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})\right)$

  $\varphi_{n,m} : \mathcal{M}_{n+m}(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{M}_m(\mathbb{C})$
- $$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto (A, D)$$

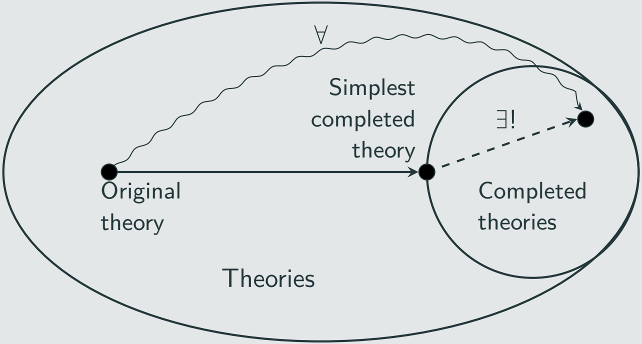
Abstract completions

Simple example: complete metric spaces

$$\begin{array}{ccc} \mathbb{Q} & \hookrightarrow & \mathbb{R} \\ & {\scriptstyle \forall f} \searrow & \vdots \; {\scriptstyle \exists! \widehat{f}} \\ & & \forall S \end{array}$$

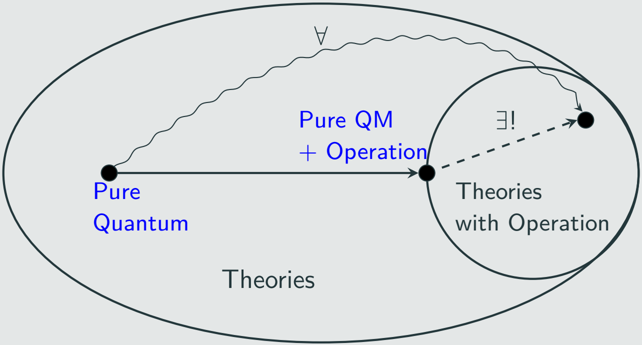
Abstract completions

Informal picture:



Abstract completions

In our case:



Completions: First example

Adding preparations

An *isometry* is either:

- a linear map $V\colon \mathbb{C}^m \to \mathbb{C}^n$ such that $\langle Va, Vb\rangle = \langle a, b\rangle$ for all a, b
- an $n \times m$ complex matrix V such that $V^* V = I$

Necessarily $m \leqslant n$ and $m = n$ precisely when an isometry is unitary.

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Necessarily $m \leqslant n$ and $m = n$ precisely when an isometry is unitary.

The bipermutative category Isometry

Is given by:

- objects: $n \in \mathbb{N}$
- morphisms: isometries $m \to n$
- $\oplus, \otimes$ as for **Unitary**

The unit N is the number zero; it is an initial object.

Isometries are a completion of unitaries

Isometries form the simplest bipermutative category containing unitaries and admitting preparation:

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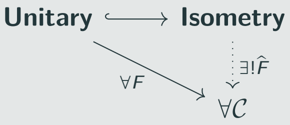
Theorem:

- $\forall\ \mathcal{C}$ bipermutative category with N initial
- $\forall\ F$: **Unitary** $\to \mathcal{C}$ strict bipermutative functor

There is a unique strict bipermutative functor $\widehat{F}$: **Isometry** $\to \mathcal{C}$ such that the following diagram commutes:



Some intuition

Adding preparation

- $\exists 0 \rightarrow n$
- $n = n + 0 \rightarrow n + p$

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Isometries

- the isometries $n \to n$ are the unitaries $n \to n$
- the isometries $1 \to n$ are the pure states

Main result: CPTP as a completion

CPTP as a completion of Isometry

Main Theorem:

- $\forall\ \mathcal{C}$ bipermutative category with I terminal
- $\forall\ F$: **Isometry** $\to \mathcal{C}$ $\oplus$-colax bipermutative functor

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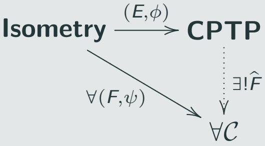

Some intuitions

Discarding:

$(\mathbf{C}, \otimes, I)$ monoidal category.

$$\begin{array}{ccc} A & \xrightarrow{!} & I \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle !} & \\ B & & \end{array}$$

$$A \otimes B \to A \otimes I = A$$

Measurement

$F(A \oplus B) \to FA \oplus FB$

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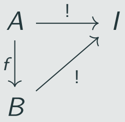

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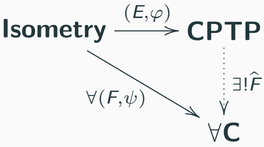
Interpretation of $\exists! \widehat{F}$

- uniqueness: no junk
- existence: no bad identifications

Remarks



- I terminal $\Rightarrow$ $\oplus$ coproduct

Remarks

$$\begin{array}{ccc} \mathbf{Isometry} & \xrightarrow{(E,\varphi)} & \mathbf{CPTP} \\ & \searrow^{\forall (F,\psi)} & \vdots\ \exists! \widehat{F} \\ & & \forall \mathbf{C} \end{array}$$

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  $\rightsquigarrow$ importance of $\oplus$-colax $\psi_{A,B} : F(A \oplus B) \to FA \oplus FB$

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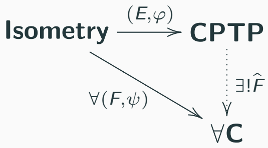

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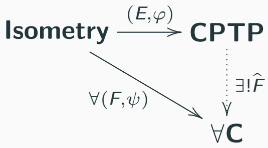

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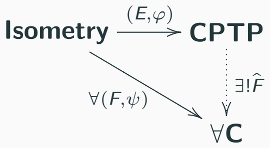

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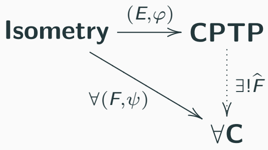
  $\rightsquigarrow$ simplified definition of **CPTP**



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- Universal properties characterise objects (essentially) uniquely
  $\rightsquigarrow$ simplified definition of **CPTP**
- Hiding does not remember the bits so measurement is needed

Conclusion

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- We introduced bipermutative categories to talk about models of Quantum theories
- We proved Quantum Channels are a canonical completion of Isometries
- Our main result also holds in the **Top**-enriched setting