Universal properties in Quantum Theory

QPL 2018, Halifax

Mathieu Huot [ENS Paris-Saclay] Sam Staton [Oxford University]

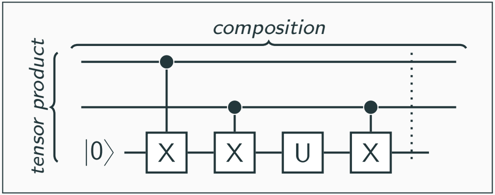
December 13, 2018

Table of contents

Introduction

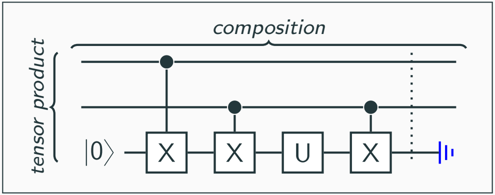
Standard point of view

- pure QM is not random, and is reversible
- pure QM does not allow discarding



Standard point of view

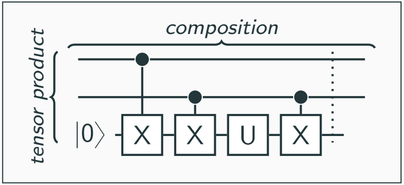
- pure QM is not random, and is reversible
- pure QM does not allow discarding
- full QM: mixed states, quantum channels



Von Neumann's model: density matrices

Pure QM

- state space $\mathbb{C}^n$
- combination of systems: $\otimes$
- ancilla (auxiliary system)
- unitary transformation U



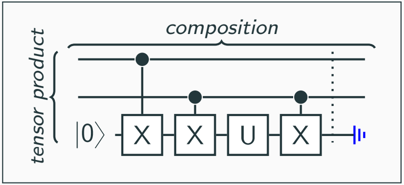
Density matrices, Completely Positive Trace Preserving (CPTP) maps

- $\mathcal{M}_n(\mathbb{C})$
- combination of systems: $\otimes$
- ancilla
- $ad_U : M \mapsto UMU^*$
  super-operator

Von Neumann's model: density matrices

Pure QM

- state space $\mathbb{C}^n$
- combination of systems: $\otimes$
- ancilla (auxiliary system)
- unitary transformation U



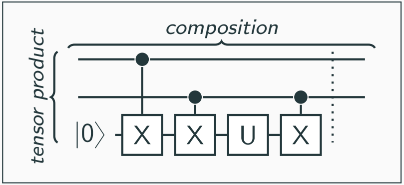
Density matrices, Completely Positive Trace Preserving (CPTP) maps

- $\mathcal{M}_n(\mathbb{C})$
- combination of systems: $\otimes$
- ancilla
- $ad_U : M \mapsto UMU^*$ super-operator
- no global phase
- allows discarding

Von Neumann's model: density matrices

Pure QM

- state space $\mathbb{C}^n$
- combination of systems: $\otimes$
- ancilla (auxiliary system)
- unitary transformation U



Density matrices, Completely Positive Trace Preserving (CPTP) maps

- $\mathcal{M}_n(\mathbb{C})$
- combination of systems: $\otimes$
- ancilla
- $ad_U : M \mapsto UMU^*$ super-operator
- no global phase
- allows discarding

Is it the right model ?

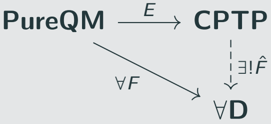
Today's presentation

Informally:
The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM, quotients global phase and allows discarding.

Today's presentation

Informally:
The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM, quotients global phase and allows discarding.

CPTP is the universal monoidal category whose unit is a terminal object:



Outline

Symmetric strict monoidal category

A symmetric monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \sigma)$ is symmetric *strict* monoidal when

- $\alpha_{A,B,C} : (A \otimes B) \otimes C = A \otimes (B \otimes C)$
- $\lambda_A : I \otimes A = A = A \otimes I$
- $\sigma_{A,B} : A \otimes B \to B \otimes A \neq id$ in general

The category Isometry (PureQM)

We define the category **Isometry** as follows:

- Objects: natural numbers n ($\mathbb{C}^n$)
- Morphisms $f : n \to m$ are linear maps $f : \mathbb{C}^n \to \mathbb{C}^m$ that are isometries: $\forall v, ||f(v)|| = ||v||$
- Composition: composition of linear maps
- $m \otimes n := mn$ and $f \otimes g$ is the usual tensor product

The category Isometry (PureQM)

We define the category **Isometry** as follows:

- Objects: natural numbers n ($\mathbb{C}^n$)
- Morphisms $f : n \to m$ are linear maps $f : \mathbb{C}^n \to \mathbb{C}^m$ that are isometries: $\forall v, ||f(v)|| = ||v||$
- Composition: composition of linear maps
- $m \otimes n := mn$ and $f \otimes g$ is the usual tensor product

Equivalently morphisms are matrices V such that $V^* V = I$ and $\otimes$ is then the Kronecker product.

The category Isometry (PureQM)

We define the category **Isometry** as follows:

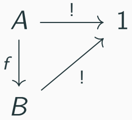
- Objects: natural numbers n ($\mathbb{C}^n$)
- Morphisms $f : n \to m$ are linear maps $f : \mathbb{C}^n \to \mathbb{C}^m$ that are isometries: $\forall v, ||f(v)|| = ||v||$
- Composition: composition of linear maps
- $m \otimes n := mn$ and $f \otimes g$ is the usual tensor product

Equivalently morphisms are matrices V such that $V^*V = I$ and $\otimes$ is then the Kronecker product.

Examples:

- the isometries $V : n \to n$ are the unitaries
- an isometry $V : 1 \to n$ is a pure state

Monoidal category with discarding:[1]
A (strict) symmetric monoidal category $(\mathbf{C}, \otimes, I)$ has discarding when the unit of the tensor product I is a terminal object.



[1] **B. Jacobs (1994): Semantics of weakening and contraction**,
D. Walker (2002): Substructural Type Systems,
P. Selinger & B. Valiron (2006): A lambda calculus for quantum computation with classical

The category CPTP (FullQM)

We define the category **CPTP** of completely positive trace preserving maps as follows:

- Objects are natural numbers n ($\mathcal{M}_n(\mathbb{C})$)
- Morphisms $f : n \to m$ are linear maps $f : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ that are completely positive and trace preserving
- Composition: composition of linear maps
- $m \otimes n := mn$ and $f \otimes g$ is again the tensor product

The category CPTP (FullQM)

We define the category **CPTP** of completely positive trace preserving maps as follows:

- Objects are natural numbers n ($\mathcal{M}_n(\mathbb{C})$)
- Morphisms $f : n \to m$ are linear maps $f : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ that are completely positive and trace preserving
- Composition: composition of linear maps
- $m \otimes n := mn$ and $f \otimes g$ is again the tensor product

CPTP has discarding

- $!_n : n \to 1$ is the trace operator
- $id_m \otimes !_n : m \otimes n \to m \otimes 1 = m$ is the partial trace operator

Symmetric strict monoidal functor

A symmetric monoidal functor F is symmetric *strict* monoidal when the isomorphisms

- $F(A) \otimes F(B) \cong F(A \otimes B)$
- $I \cong F(I)$

are identities.

Symmetric strict monoidal functor

A symmetric monoidal functor F is symmetric *strict* monoidal when the isomorphisms

- $F(A) \otimes F(B) \cong F(A \otimes B)$
- $I \cong F(I)$

are identities.

E : Isometry $\to$ CPTP

- $E(n) := n$
- $E(V) := ad_V : M \mapsto VMV^*$

Outline

Main theorem

The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM, quotients global phase and allows discarding.

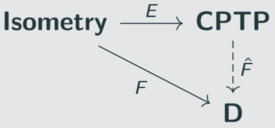
Main theorem

The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM, quotients global phase and allows discarding.

Theorem: universality of CPTP

- $\forall$ $\mathbf{D}$ strict symmetric monoidal category with discarding
- $\forall$ $F : \mathbf{Isometry} \to \mathbf{D}$ symmetric strict monoidal functor

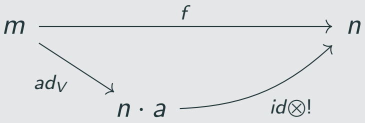
There is a unique symmetric monoidal functor $\hat{F} : \mathbf{CPTP} \to \mathbf{D}$ such that:



Proof: key lemma

Stinespring theorem[2]

For every CPTP f there is a pair (V, a) such that:

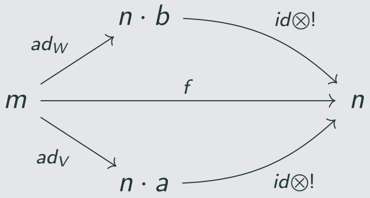


[2]**B. Coecke & A. Kissinger (2017): Picturing Quantum Processes. CUP.**.

Proof: key lemma

Stinespring theorem[2]

For every CPTP f there is a pair (V, a) such that:

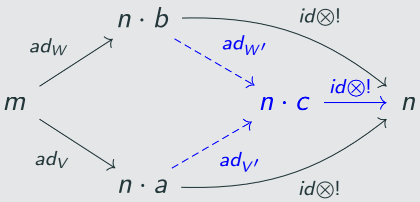


[2]**B. Coecke & A. Kissinger (2017): Picturing Quantum Processes. CUP.**.

Proof: key lemma

Stinespring theorem[2]

For every CPTP f there is a pair (V, a) such that:

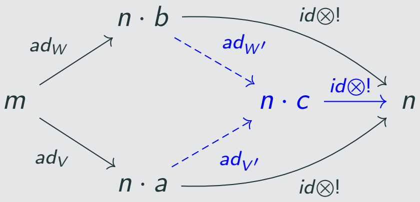


[2]**B. Coecke & A. Kissinger (2017): Picturing Quantum Processes. CUP.**.

Proof: key lemma

Stinespring theorem[2]

For every CPTP f there is a pair (V, a) such that:



(V, a) is called a dilation for f.

[2] **B. Coecke & A. Kissinger (2017): Picturing Quantum Processes. CUP.**.

Proof: uniqueness

If any symmetric monoidal functor $\hat{F}$ is going to make diagram commute then it must be defined as

- $\hat{F}(n) \stackrel{\text{def}}{=} F(n)$ as E is identity on objects
- If $\begin{array}{ccc} n & \xrightarrow{f} & m \\ & {\scriptstyle ad_V} \searrow \quad \nearrow {\scriptstyle id \otimes !} & \\ & n \cdot a & \end{array}$ then $\begin{array}{ccc} n & \xrightarrow{\hat{F}(f)} & m \\ & {\scriptstyle \hat{F}(ad_V)} \searrow \quad \nearrow {\scriptstyle \hat{F}(id \otimes !)} & \\ & n \cdot a & \end{array}$

Proof: uniqueness

If any symmetric monoidal functor $\hat{F}$ is going to make diagram commute then it must be defined as

- $\hat{F}(n) \stackrel{\text{def}}{=} F(n)$ as E is identity on objects
- If $\begin{array}{ccc} n & \xrightarrow{\quad f \quad} & m \\ {\scriptstyle ad_V} \searrow & & \nearrow {\scriptstyle id \otimes !} \\ & n \cdot a & \end{array}$ then $\begin{array}{ccc} n & \xrightarrow{\quad \hat{F}(f) \quad} & m \\ {\scriptstyle F(V)} \searrow & & \nearrow {\scriptstyle (id \otimes !)} \\ & n \cdot a & \end{array}$

Proof: well-definedness

Only choice :

- $\hat{F}(n) \stackrel{\text{def}}{=} F(n)$
- $\hat{F}((id \otimes !) \circ ad_V) \stackrel{\text{def}}{=} (id \otimes !) \circ F(V)$

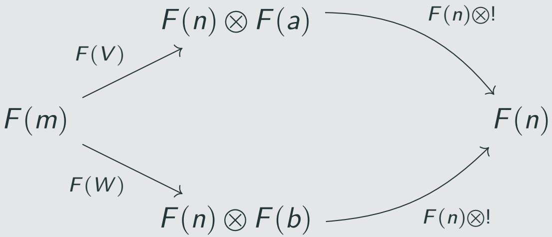
Proof: well-definedness

Only choice :

- $\hat{F}(n) \stackrel{\text{def}}{=} F(n)$
- $\hat{F}((id \otimes !) \circ ad_V) \stackrel{\text{def}}{=} (id \otimes !) \circ F(V)$

Independence of the choice of dilation (V, a)

Given (W, b) another dilation,



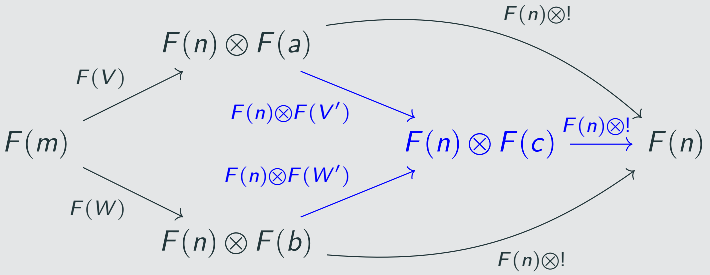
Proof: well-definedness

Only choice :

- $\hat{F}(n) \stackrel{\text{def}}{=} F(n)$
- $\hat{F}((id \otimes !) \circ ad_V) \stackrel{\text{def}}{=} (id \otimes !) \circ F(V)$

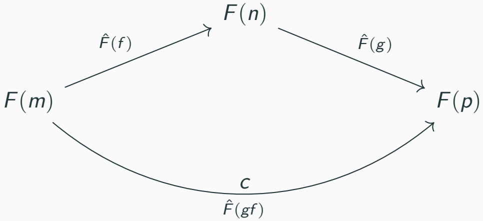
Independence of the choice of dilation (V, a)

Given (W, b) another dilation,Stinespring theorem guarantees there is a triple (c, V', W') such that:



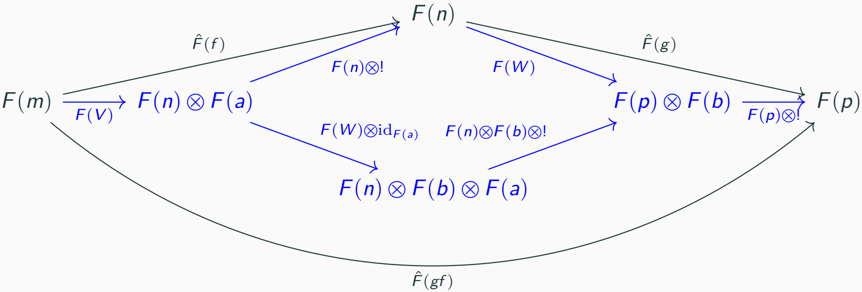
Proof: functoriality

- Identity: dilation $(id_n, 1)$
- Composition: if (V, a) is a dilation of $f : m \to n$ and (W, b) is a dilation of $g : n \to p$,



Proof: functoriality

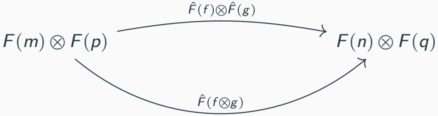
- Identity: dilation $(id_n, 1)$
- Composition: if (V, a) is a dilation of $f : m \to n$ and (W, b) is a dilation of $g : n \to p$, then $((W \otimes \mathrm{id}_a)V, b \otimes a)$ is a dilation of gf.



Proof: monoidal functor

- (V, a) is a dilation of $f : m \to n$
- (W, b) is a dilation of $g : p \to q$

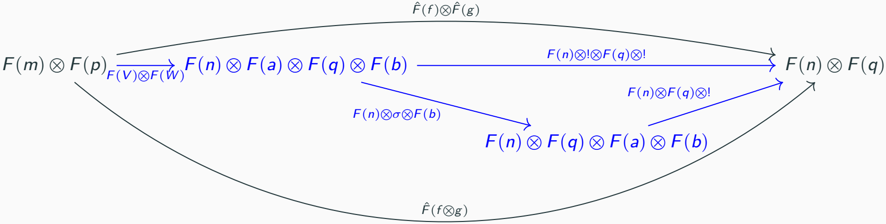
Then $((\mathrm{id}_m \otimes \sigma \otimes \mathrm{id}_p) \circ (V \otimes W), a \otimes b)$ is a dilation of $f \otimes g$.



Proof: monoidal functor

- (V, a) is a dilation of $f : m \to n$
- (W, b) is a dilation of $g : p \to q$

Then $((\mathrm{id}_m \otimes \sigma \otimes \mathrm{id}_p) \circ (V \otimes W), a \otimes b)$ is a dilation of $f \otimes g$.



Outline

B. Coecke (2006): Interpretation of the universality[3]

- not *ad hoc*
- new definition for CPTP
- relies on Stinespring theorem (purification uniqueness)

$$\begin{array}{ccc} \textbf{Isometry} & \xrightarrow{E} & \textbf{CPTP} \\ & {\scriptstyle \forall F} \searrow & \Big\downarrow {\scriptstyle \exists! \hat{F}} \\ & & \forall \textbf{D} \end{array}$$

[3] **Axiomatic description of mixed states from Selinger“leC –“textquoteright ˝s CPM-constr**
O. Cunningham & C. Heunen (2015): Axiomatizing complete positivity.

Affine reflection[4]

- $\mathfrak{SMCat}$: category of (small) symmetric monoidal categories and symmetric monoidal functors
- $\mathfrak{AMCat}$: category of (small) monoidal categories for which the unit is terminal and symmetric monoidal functors. Affine≅With discarding

The full and faithful embedding $\mathfrak{AMCat} \to \mathfrak{SMCat}$ has a left adjoint $L : \mathfrak{SMCat} \to \mathfrak{AMCat}$.

In other words, $\mathfrak{AMCat}$ is a reflective subcategory of $\mathfrak{SMCat}$.

[4] **C. Hermida & R.D. Tennent (2009): Monoidal Indeterminates and Categories of Possible**

Affine reflection[4]

- $\mathfrak{SMCat}$: category of (small) symmetric monoidal categories and symmetric monoidal functors
- $\mathfrak{AMCat}$: category of (small) monoidal categories for which the unit is terminal and symmetric monoidal functors. Affine≅With discarding

The full and faithful embedding $\mathfrak{AMCat} \to \mathfrak{SMCat}$ has a left adjoint $L : \mathfrak{SMCat} \to \mathfrak{AMCat}$.

In other words, $\mathfrak{AMCat}$ is a reflective subcategory of $\mathfrak{SMCat}$.

Corollary:

The symmetric monoidal category of CPTP maps is the affine reflection of the symmetric monoidal category of isometries:

$$L(\mathbf{Isometry}) \cong \mathbf{CPTP}$$

[4] C. Hermida & R.D. Tennent (2009): Monoidal Indeterminates and Categories of Possible

Affine Reflection for Injection

Let **Injection** be the category whose objects are natural numbers and morphisms $f : n \to m$ are injective functions $f : \underline{n} \to \underline{m}$ where $\underline{n} := \{0, \ldots, n-1\}$.

Affine Reflection for Injection

Let **Injection** be the category whose objects are natural numbers and morphisms $f : n \to m$ are injective functions $f : \underline{n} \to \underline{m}$ where $\underline{n} := \{0, \ldots, n-1\}$.

If $f : n \to m$ is an injection, then we define the isometry matrix V_f by $V_f(e_i) = e_{f(i)}$. If $f : n \to n$ is a bijection then V_f is a permutation matrix. This gives a monoidal embedding **Injection** $\to$ **Isometry**.

Affine Reflection for Injection

Let **Injection** be the category whose objects are natural numbers and morphisms $f : n \to m$ are injective functions $f : \underline{n} \to \underline{m}$ where $\underline{n} := \{0, \dots, n-1\}$.

If $f : n \to m$ is an injection, then we define the isometry matrix V_f by $V_f(e_i) = e_{f(i)}$. If $f : n \to n$ is a bijection then V_f is a permutation matrix. This gives a monoidal embedding **Injection** $\to$ **Isometry**.

CPTP has a subcategory **ICPTP** for which morphisms have dilations of the form (V_f, a).

Affine Reflection for Injection

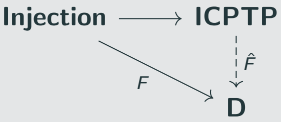
Let **Injection** be the category whose objects are natural numbers and morphisms $f : n \to m$ are injective functions $f : \underline{n} \to \underline{m}$ where $\underline{n} := \{0, \ldots, n-1\}$.

If $f : n \to m$ is an injection, then we define the isometry matrix V_f by $V_f(e_i) = e_{f(i)}$. If $f : n \to n$ is a bijection then V_f is a permutation matrix. This gives a monoidal embedding **Injection** $\to$ **Isometry**.

CPTP has a subcategory **ICPTP** for which morphisms have dilations of the form (V_f, a).

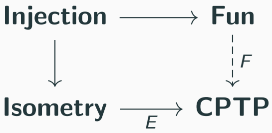
ICPTP is the affine completion of Injection

For every symmetric strict monoidal functor F, there is a unique symmetric monoidal functor $\hat{F}$: **ICPTP** $\to$ **D** such that:



Fun is not the affine reflection of Injection

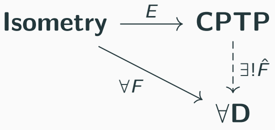
There is no strict monoidal functor F making the following diagram commute:



We have a simple counter-example.

Summary

- CPTP are canonical (universal property)
- Motivated by physics arguments
- Link to affine reflections



Some directions

- Consider the other monoidal structure $\oplus$
- Topology on the unitaries
- More affine completions

Some directions

- Consider the other monoidal structure $\oplus$
- Topology on the unitaries
- More affine completions

Isometry is the co-affine reflection of Unitary for $\oplus$: