Universal properties in Quantum Theory

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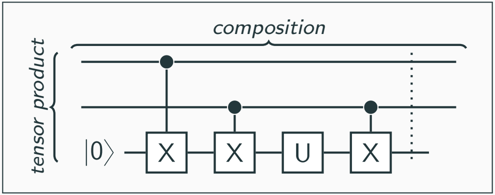
Introduction

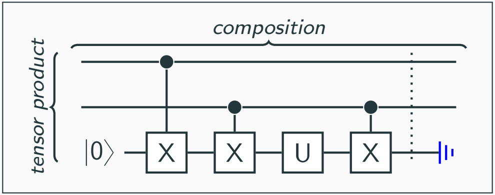
Standard point of view

- pure QM is not random, and is reversible
- pure QM does not allow discarding



Standard point of view

- pure QM is not random, and is reversible
- pure QM does not allow discarding
- full QM: mixed states, quantum channels



Von Neumann's model: density matrices

Pure QM (+ ancillas)

- state space $\mathbb{C}^n$
- combination of systems: $\otimes$
- ancilla (auxiliary system)
- unitary transformation U

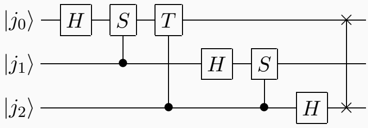


Figure 1: Quantum Fourier transform on three qubits

Completely Positive Trace Preserving (CPTP) maps

- $\mathcal{M}_n(\mathbb{C})$
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- $ad_U : M \mapsto UMU^*$ super-operator

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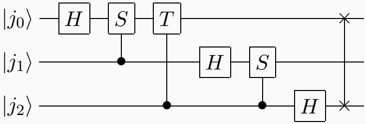

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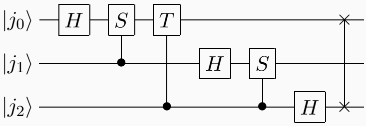

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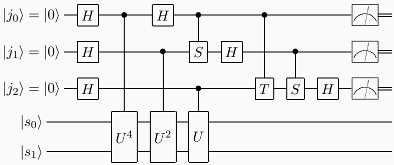

Figure 2: Three-qubit phase estimation circuit with QFT and controlled-U

Today's presentation

Informally:
The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM with ancillas, quotients global phase and allows discarding.

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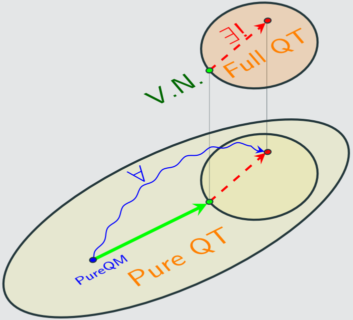
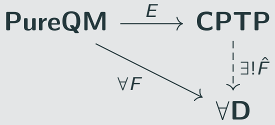
The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM with ancillas, quotients global phase and allows discarding.

CPTP is the universal monoidal category on PureQM whose unit is a terminal object:





Outline

Symmetric strict monoidal category

A symmetric monoidal category $(\mathcal{C}, \otimes, I, \alpha, \lambda, \sigma)$ is symmetric *strict* monoidal when

- $\alpha_{A,B,C} : (A \otimes B) \otimes C = A \otimes (B \otimes C)$
- $\lambda_A : I \otimes A = A = A \otimes I$
- $\sigma_{A,B} : A \otimes B \to B \otimes A \neq id$ in general

The category Isometry (PureQM)

We define the category **Isometry** as follows:

- Objects: natural numbers n ($\mathbb{C}^n$)
- Morphisms $f : n \to m$ are linear maps $f : \mathbb{C}^n \to \mathbb{C}^m$ that are isometries: $\forall v, ||f(v)|| = ||v||$
- Composition: composition of linear maps
- $m \otimes n := mn$ and $f \otimes g$ is the usual tensor product

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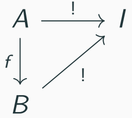
Examples:

- the isometries $V : n \to n$ are the unitaries
- an isometry $V : 1 \to n$ is a pure state

Discarding

Monoidal category with discarding:[1]
A (strict) symmetric monoidal category $(\mathbf{C}, \otimes, I)$ has discarding when the unit of the tensor product I is a terminal object.



[1]**B. Jacobs (1994): Semantics of weakening and contraction**,
D. Walker (2002): Substructural Type Systems,
P. Selinger & B. Valiron (2006): A lambda calculus for quantum computation with classical

The category CPTP (FullQM)

We define the category **CPTP** of completely positive trace preserving maps as follows:

- Objects are natural numbers n ($\mathcal{M}_n(\mathbb{C})$)
- Morphisms $f : n \to m$ are linear maps $f : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ that are completely positive and trace preserving (quantum channels)
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CPTP has discarding

- $!_n : n \to 1$ is the trace operator
- $id_m \otimes !_n : m \otimes n \to m \otimes 1 = m$ is the partial trace operator

Symmetric strict monoidal functor

A symmetric monoidal functor F is symmetric *strict* monoidal when the isomorphisms

- $F(A) \otimes F(B) \cong F(A \otimes B)$
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E : Isometry $\to$ CPTP

- $E(n) := n$
- $E(V) := ad_V : M \mapsto VMV^*$

Outline

Main theorem

The category of Completely Positive Trace Preserving (CPTP) is the simplest category that interprets PureQM with ancillas, quotients global phase and allows discarding.

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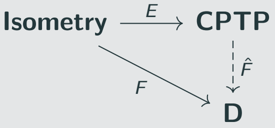
Theorem: universality of CPTP

- $\forall$ **D** strict symmetric monoidal category with discarding
- $\forall$ F : **Isometry** $\to$ **D** symmetric strict monoidal functor

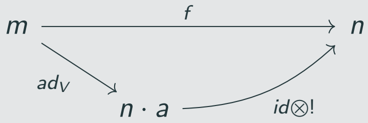
There is a unique symmetric strict monoidal functor $\hat{F}$: **CPTP** $\to$ **D** such that:



Proof: key lemma

Stinespring's theorem[2]

For every CPTP f there is a pair (V, a) such that:



[2] **B. Coecke & A. Kissinger (2017): Picturing Quantum Processes. CUP.**.

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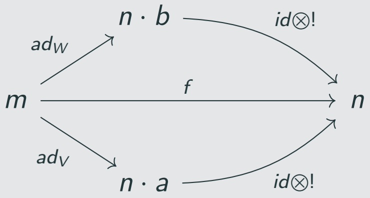

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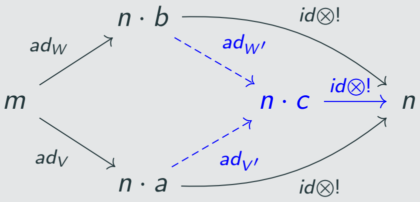

(V, a) is called a dilation for f.

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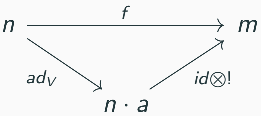
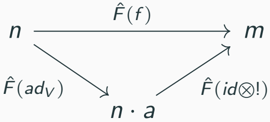
Proof: uniqueness

If any symmetric monoidal functor $\hat{F}$ is going to make diagram commute then it must be defined as

- $\hat{F}(n) \stackrel{\text{def}}{=} F(n)$ as E is identity on objects
- If 

then 

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- If $\begin{array}{ccc} n & \xrightarrow{f} & m \\ & {\scriptstyle ad_V} \searrow \quad \nearrow {\scriptstyle id \otimes !} & \\ & n \cdot a & \end{array}$ then $\begin{array}{ccc} n & \xrightarrow{\hat{F}(f)} & m \\ & {\scriptstyle F(V)} \searrow \quad \nearrow {\scriptstyle (id \otimes !)} & \\ & n \cdot a & \end{array}$

Proof: well-definedness

Only choice :

- $\hat{F}(n) \stackrel{\mathrm{def}}{=} F(n)$
- $\hat{F}((id \otimes !) \circ ad_V) \stackrel{\mathrm{def}}{=} (id \otimes !) \circ F(V)$

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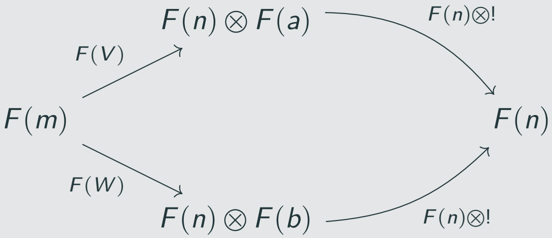
Independence of the choice of dilation (V, a)

Given (W, b) another dilation,



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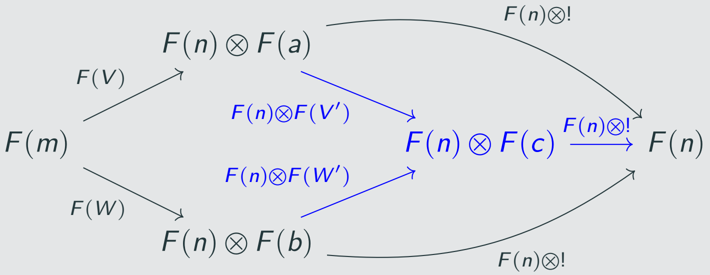
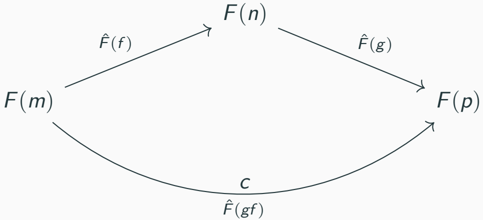
Independence of the choice of dilation (V, a)

Given (W, b) another dilation,Stinespring theorem guarantees there is a triple (c, V', W') such that:



Proof: functoriality

- Identity: dilation $(id_n, 1)$
- Composition: if (V, a) is a dilation of $f : m \to n$ and (W, b) is a dilation of $g : n \to p$,



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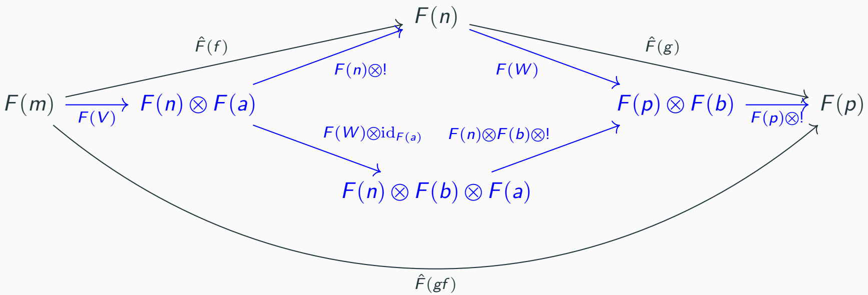

Proof: monoidal functor

- (V, a) is a dilation of $f : m \to n$
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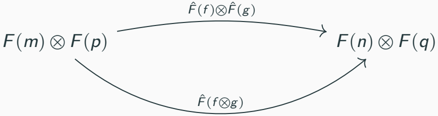
Then $((\mathrm{id}_m \otimes \sigma \otimes \mathrm{id}_p) \circ (V \otimes W), a \otimes b)$ is a dilation of $f \otimes g$.



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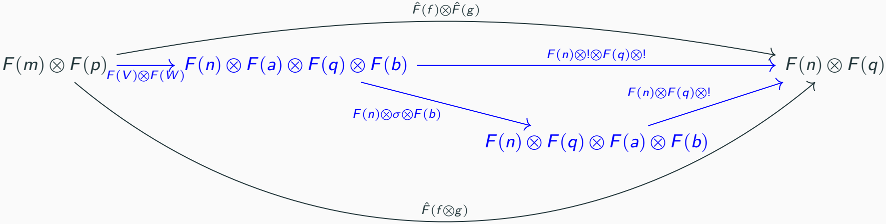
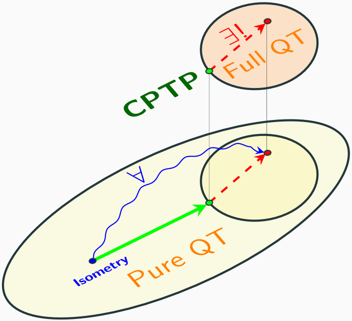
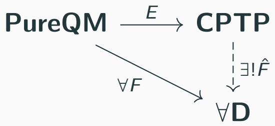


Interpretation of the universality[3]

- foundational justification for the model
- new definition for CPTP
- relies on Stinespring theorem (purification uniqueness)





[3]B. Coecke (2006): Axiomatic description of mixed states from Selinger's CPM-constructi
O. Cunningham & C. Heunen (2015): Axiomatizing complete positivity.

Outline

Affine reflection[4]

- $\mathfrak{SMCat}$: category of (small) symmetric strict monoidal categories and symmetric monoidal functors
- $\mathfrak{AMCat}$: category of (small) symmetric strict monoidal categories for which the unit is terminal and symmetric monoidal functors.
  Affine$\cong$With discarding

The full and faithful embedding $\mathfrak{AMCat} \to \mathfrak{SMCat}$ has a left adjoint $L : \mathfrak{SMCat} \to \mathfrak{AMCat}$.

In other words, $\mathfrak{AMCat}$ is a reflective subcategory of $\mathfrak{SMCat}$.

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Corollary:

The symmetric monoidal category of CPTP maps is the affine reflection of the symmetric monoidal category of isometries:

$$L(\mathbf{Isometry}) \cong \mathbf{CPTP}$$

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PROPs

A PROP is a symmetric strict monoidal category generated by one object.

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PROPs: $\mathbf{Isometry}_2$ and $\mathbf{CPTP}_2$

- $\mathbf{Isometry}_2$: full subcategory of **Isometry** whose objects are powers of 2
- $\mathbf{CPTP}_2$: full subcategory of **CPTP** whose objects are powers of 2
- $E : \mathbf{Isometry} \to \mathbf{CPTP}$ restricts to a symmetric strict monoidal functor $E_2 : \mathbf{Isometry}_2 \to \mathbf{CPTP}_2$

Universality of $CPTP_2$

Theorem: universality of $CPTP_2$

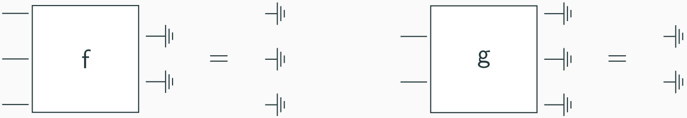
$$\begin{array}{ccc} \mathbf{Isometry}_2 & \xrightarrow{E_2} & \mathbf{CPTP}_2 \\ & \searrow^{\forall F} & \Big\downarrow{\scriptstyle \exists! \hat{F}} \\ & & \forall \mathbf{D} \end{array}$$

where:

- $F, \widehat{F}$ are symmetric strict monoidal functors
- $\mathbf{D}$ is a symmetric strict monoidal category

A relation to quantum circuits

Affine reflection of a PROP:
When **D** is a PROP, the affine reflection $L(\mathbf{D})$ is a PROP, presented by one generating morphism ⏚ $: 1 \to 0$ and equations of the form:



and so on.

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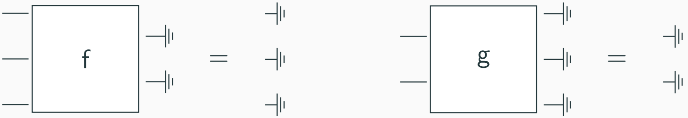

and so on.

Consequence:

$\mathbf{CPTP}_2$ is obtained by freely adding discarding to $\mathbf{Isometry}_2$

Outline

Enriched categories

A functor $F : \mathcal{C} \to \mathcal{D}$ between (locally small) categories $\mathcal{C}, \mathcal{D}$ induces $\forall A, B \in Obj(\mathcal{C})$ a **Set** function $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$.

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$\mathcal{C}(A, B), \mathcal{D}(FA, FB)$ and $F_{A,B}$ are equiped with the structure from the Cartesian monoidal category **Set**.

More generally they could be equiped with the structure of a monoidal category $(\mathcal{V}, \otimes, I)$, such as **Top** and **Met**.

Examples:

- **Top**: topological spaces and continuous maps, with Cartesian product
- **Met**: metric spaces and short maps, with $A \otimes B := A \times B$ and $d_{A \otimes B} = d_A + d_B$

Isometry and CPTP as enriched categories

Linear functions $f : V \to W$ can be equiped with the operator norm $||f||_{op} := sup_{||v||_V=1}||f(v)||_W$.

Isometry and CPTP as enriched categories

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This gives a norm and hence a distance on isometries, and also on CPTP maps by $d(f,g) := ||f - g||_{op}$.

The metric induces a topology on isometries and on CPTP maps.

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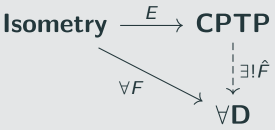
We can therefore see **Isometry** and **CPTP** as enriched categories, and E as an enriched functor, both over **Top** or over **Met**.

Enriched completion theorem

Theorem: universality of CPTP in the enriched setting



where:

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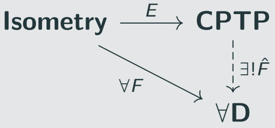

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None of the theorems is trivially deduced from the others.

Considering the second tensor product

$\oplus$ is a second tensor product on vector spaces and linear maps. It restricts to a tensor on **Isometry** and to a small extension $\mathbf{CPTP}'$ of **CPTP**.

There is a distributivity law $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$. $\oplus$ is responsible for entanglement — e.g. $(Id_2 \oplus X)$ is the controlled-not operator on one qubit —

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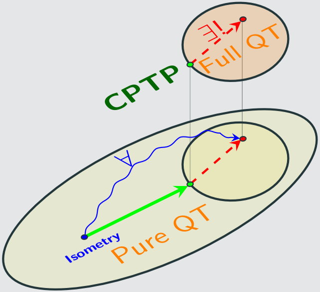
There is a distributivity law $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$. $\oplus$ is responsible for entanglement — e.g. $(Id_2 \oplus X)$ is the controlled-not operator on one qubit —

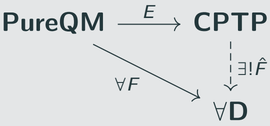
CPTP$'$ is a completion of Isometry

In this setting with two tensors and a distributive law, **CPTP**$'$ is a lax completion of **Isometry**, where the lax morphism $\mathcal{M}_{n+n}(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}) \oplus \mathcal{M}_n(\mathbb{C})$ gives measurement.

Summary and conclusion: CPTP is canonical

CPTP is the universal monoidal category on Isometry whose unit is a terminal object:





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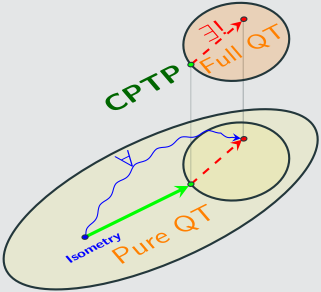

$$\begin{array}{ccc} \textbf{PureQM} & \xrightarrow{E} & \textbf{CPTP} \\ & \underset{\forall F}{\searrow} & \downarrow{\scriptstyle \exists! \hat{F}} \\ & & \forall \textbf{D} \end{array}$$

- In the broader context of affine reflections
- Theorem for underlying PROPs $\textbf{Isometry}_2 \to \textbf{CPTP}_2$
- Theorems in the topological and the metric enriched cases
- Added the second tensor product $\oplus$ to recover bits, measurement