1. SOLOW SWAN MODEL WITH NO TECHNOLOGICAL GROWTH

K – CAPITAL STOCK L – LABOUR FORCE

$\hat{k} = \frac{K}{L}$ – CAPITAL PER WORKER

$F(K,L) = AK^{\alpha}L^{1-\alpha}$ – C.R.S. COBB-DOUGLASS PRODUCTION FUNCTION, A IS A CONSTANT

$\dot{L} = nL$ – CHANGE IN LABOUR FORCE

s – SAVINGS RATE (EXOGENOUS)

$\dot{K} = sAK^{\alpha}L^{1-\alpha} - \delta K$ – CAPITAL ACCUMULATION EQUATION

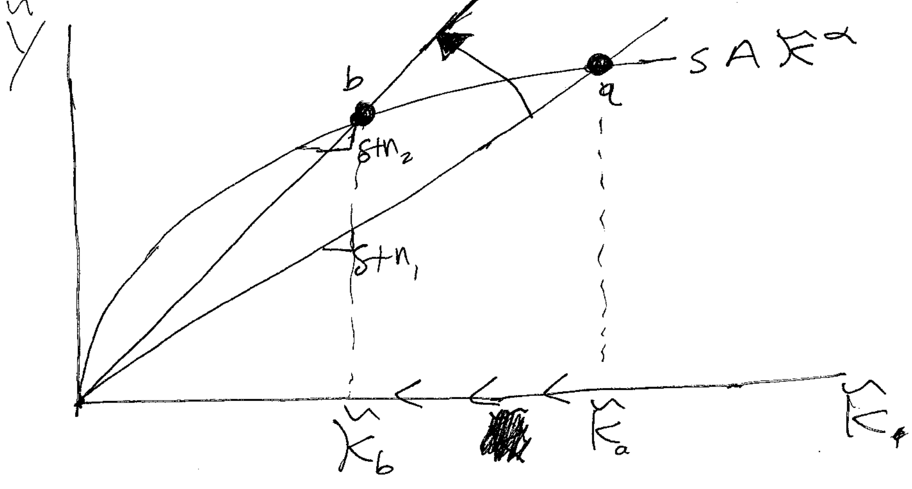
STEADY STATE IS WHERE $\dot{\hat{k}} = \frac{L\dot{K} - K\dot{L}}{L^2} = 0$

$$\dot{\hat{k}} = \frac{LsAK^{\alpha}L^{1-\alpha} - \cancel{\delta K} - KnL}{L^2} = sA\left(\frac{K}{L}\right)^{\alpha} - \left(\frac{K}{L}\right)n - \left(\frac{K}{L}\right)\delta$$

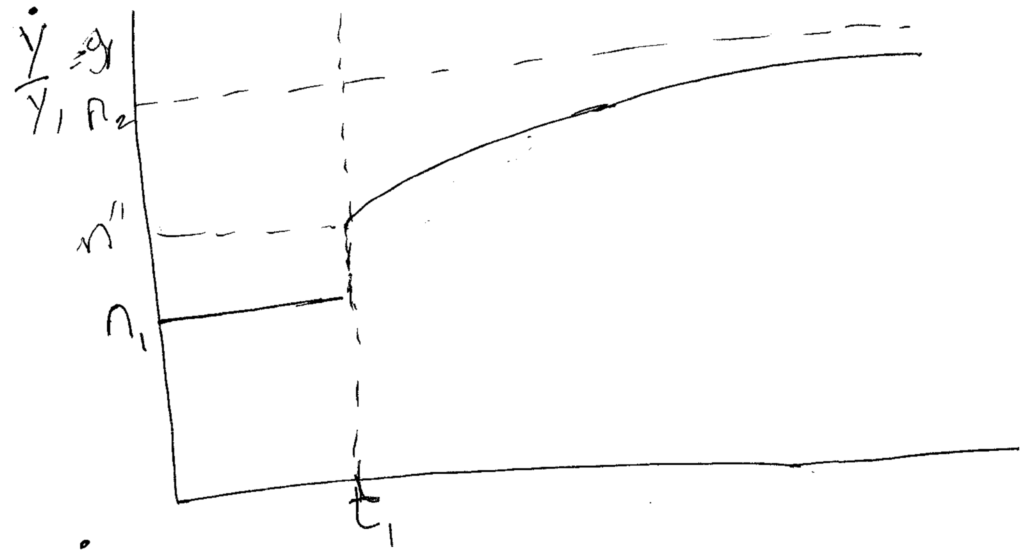
$$\Rightarrow \dot{\hat{k}} = sA(\hat{k})^{\alpha} - (n+\delta)(\hat{k})$$

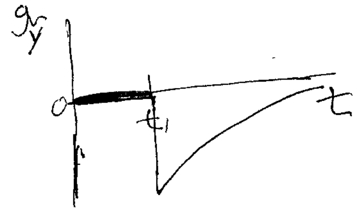
IN STEADY STATE $\hat{k} = \frac{K}{L}$ IS CONSTANT. OUTPUT PER WORKER IS THEN $\frac{AK^{\alpha}L^{1-\alpha}}{L} = A\left(\frac{K}{L}\right)^{\alpha} = A(\hat{k})^{\alpha}$ SO THIS IS ALSO CONSTANT. CONSUMPTION PER WORKER IS THEN $(1-s)A(\hat{k})^{\alpha}$ SO THIS IS ALSO CONSTANT. SINCE OUTPUT PER WORKER IS CONSTANT, STEADY ~~STATE~~ OUTPUT GROWS AT n (THE SAME RATE AS THE LABOUR FORCE). SO THE ECONOMY DOES GROW IN THE STEADY STATE, BUT NOT IN PER CAPITA TERMS DUE TO DECLINING MPK AND FIXED TECHNOLOGY PARAMETER A.

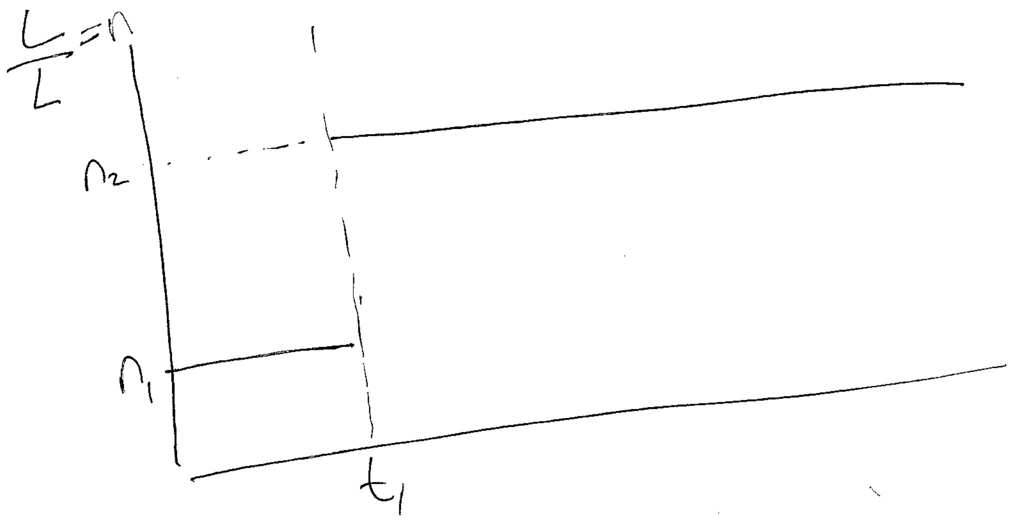
(i) IF n INCREASES: (FROM n_1 TO n_2)

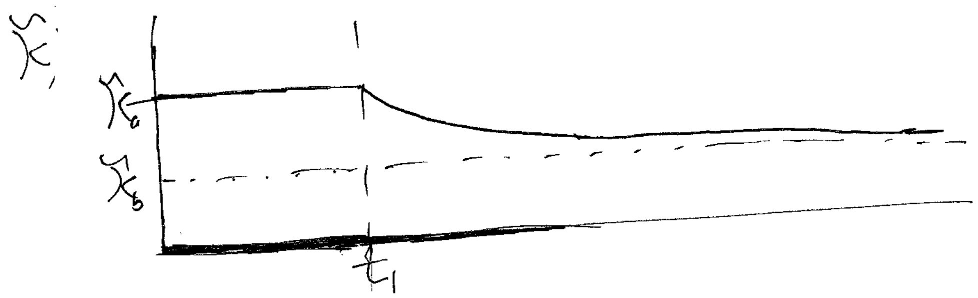


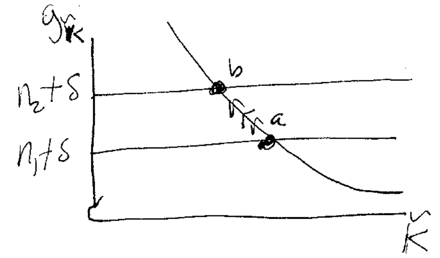
INITIALLY WE ARE AT STEADY STATE a WITH LABOUR, CAPITAL AND OUTPUT ALL GROWING AT RATE n_1. WHEN POPULATION GROWTH RATE INCREASES, SAVINGS ARE NO LONGER SUFFICIENT TO KEEP CAPITAL PER WORKER CONSTANT. SO, $\hat{k}$ DECREASES, THUS DECREASING OUTPUT PER WORKER AS WELL. THIS CONTINUES UNTIL $\hat{k}$ REACHES $\hat{k}_b$, AT WHICH POINT CAPITAL AND OUTPUT GROW AT RATE n_2.





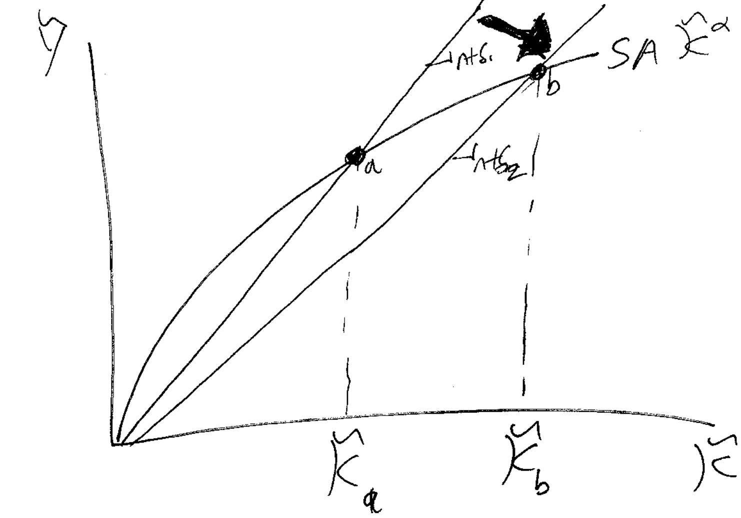


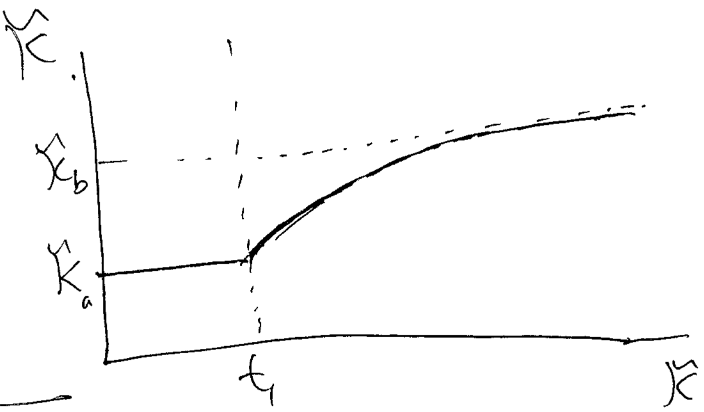


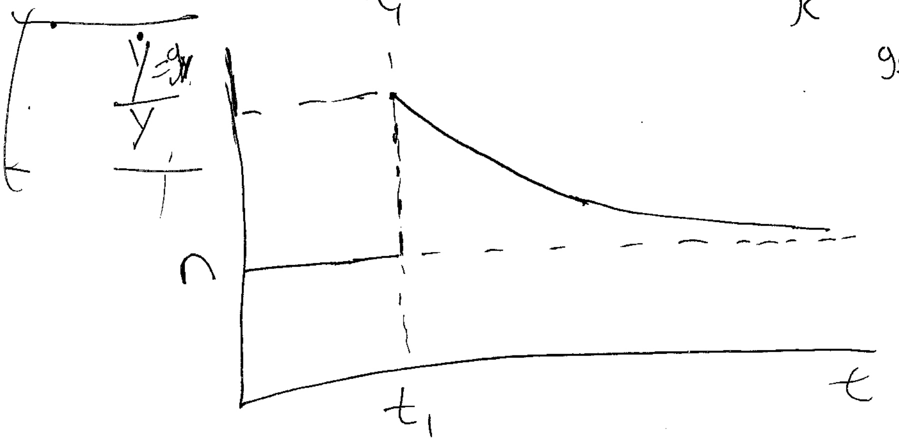


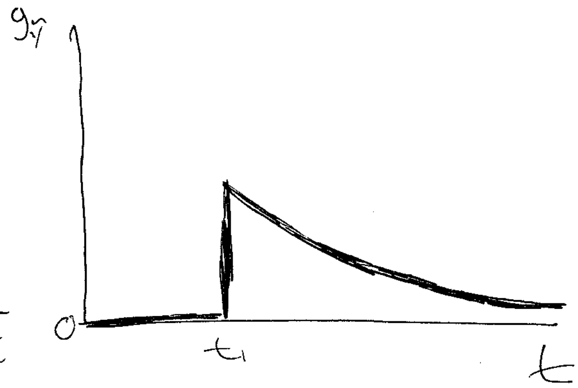
NOTE: ALTHOUGH RISE IN n FROM n_1 TO n_2 CAUSES CAPITAL AND OUTPUT PER WORKER TO DECREASE, IT CAUSES GROWTH RATE OF CAPITAL AND OUTPUT TO STEADILY INCREASE TO THE NEW STEADY STATE RATE GROWTH RATE n_2.

(ii) IF δ FALLS FROM δ_1 TO δ_2:

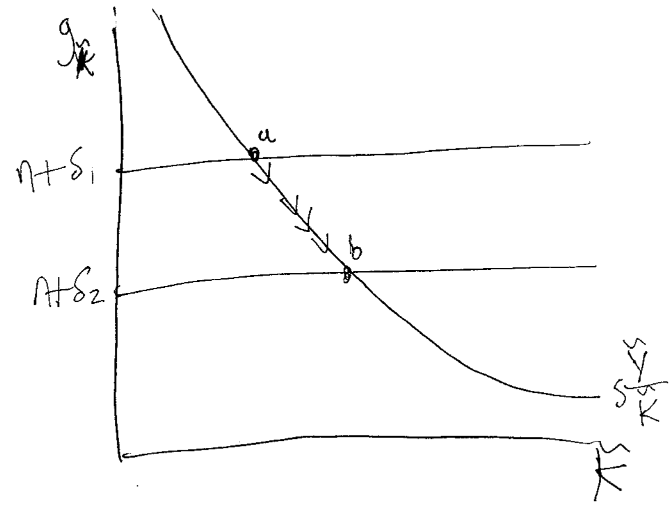








HERE THE SHORT RUN EFFECT OF THE DECREASED DEPRECIATION RATE IS TO BOOST THE GROWTH RATE OF CAPITAL AND OUTPUT ABOVE n, BUT GRADUALLY THE GROWTH RATE RETURNS TO n IN THE LONG RUN.



2. AS ALREADY SHOWN, WITH EXOGENOUS SAVINGS CONSUMPTION PER WORKER IS GIVEN BY $(1-s)A(\hat{k})^{\alpha}$ WHERE $\hat{k}$ IS SUCH THAT $\dot{\hat{k}} = sA\hat{k}^{\alpha} - (n+\delta)\hat{k} = 0$

$$\Rightarrow sA(\hat{k})^{\alpha} = (n+\delta)\hat{k} \Rightarrow s = \frac{(n+\delta)\hat{k}}{A\hat{k}^{\alpha}}$$

$$\Rightarrow (\hat{k})^{1-\alpha} = \frac{sA}{n+\delta} \Rightarrow \hat{k} = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

SUBBING THIS IN GIVES $\hat{c} = \frac{c}{L} = (1-s)A\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$

$$\Rightarrow \hat{c} = A^{\frac{1-\alpha}{1-\alpha}+\frac{\alpha}{1-\alpha}}\left(\frac{1}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} s^{\frac{\alpha}{1-\alpha}}(1-s)$$

$$\Rightarrow \hat{c} = A^{\frac{1}{1-\alpha}}\left(\frac{1}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} s^{\frac{\alpha}{1-\alpha}}(1-s)$$

NOW WE CAN FIND THE s WHICH MAXIMISES $\hat{c}$:

$$\frac{d\hat{c}}{ds} = A^{\frac{1}{1-\alpha}}\left(\frac{1}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\alpha}{1-\alpha}s^{\frac{\alpha}{1-\alpha}-1}(1-s) - s^{\frac{\alpha}{1-\alpha}}\right) = 0$$

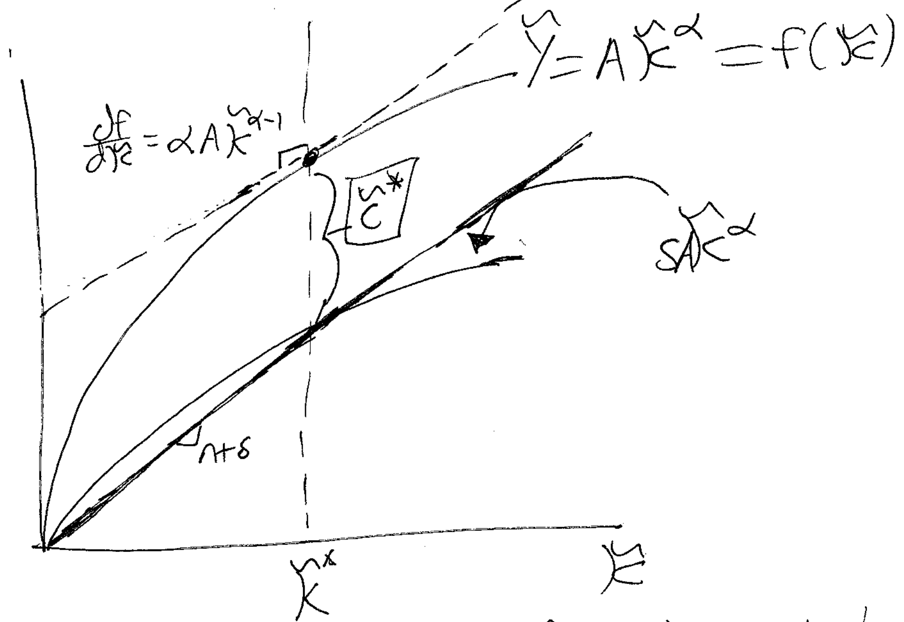
$$\Rightarrow \frac{d\hat{c}}{ds} = A^{\frac{2-\alpha}{1-\alpha}}\left(\frac{1}{n+\delta}\right)^{\frac{1}{1-\alpha}}\left(s^{\frac{\alpha}{1-\alpha}}\right)\left(\left(\frac{\alpha}{1-\alpha}\right)\frac{1-s}{s} - 1\right) = 0$$

$$\Rightarrow \alpha(1-s^{*}) = (1-\alpha)s^{*} \Rightarrow s^{*} = \alpha$$

NOTE THAT SINCE $0<\alpha<1$, $0<s^{*}<1$.

INTUITION – THE HIGHER THE WEIGHTING ON CAPITAL IN THE PRODUCTION THE HIGHER THE GOLDEN RULE SAVINGS RATE.

GRAPHICALLY, s^{*} MAXIMIZES THE DIFFERENCE BETWEEN THE OUTPUT AT SAVINGS CURVES, AT THE STEADY STATE:



The gradient of the $\hat{y}$ line and the $(n+\delta)\hat{k}$ line must be the same, so we need $n+\delta = \frac{\alpha A\hat{k}^{\alpha}}{\hat{k}}$ AND $(n+\delta)(\hat{k}) = sA\hat{k}^{\alpha}$

GENERALLY: $n+\delta = \frac{df}{d\hat{k}}$ AND $(n+\delta)(\hat{k}) = sf(\hat{k})$

2. WITH ENDOGENOUS SAVINGS, WE MAINTAIN THE CAPITAL ACCUMULATION IDENTITY EXCEPT THAT CONSUMPTION AND SAVINGS ARE NOW DETERMINED ENDOGENOUSLY. SO: $\dot{K} = AF(K,L) - C - \delta K$

$$\Rightarrow \dot{\hat{k}} = \frac{L\dot{K} - K\dot{L}}{L^2} = \frac{AF(K,L)}{L} - \frac{C}{L} - \delta\frac{K}{L} - \left(\frac{K}{L}\right)\left(\frac{\dot{L}}{L}\right)$$

$$\Rightarrow \dot{\hat{k}} = Af(\hat{k}) - \hat{C} - \delta\hat{k} - n\hat{k}$$

WE THEN NEED TO DERIVE AN EXPRESSION FOR $\dot{\hat{C}}$ USING THE <u>EULER EQUATION</u> DERIVED FROM A SINGLE CONSUMER'S LIFETIME INTERTEMPORAL OPTIMISATION DECISION:

$$L = \sum_{i=1}^{n}\left[u(C_i)\left(\frac{1}{1+\rho}\right)^i\right] - \lambda\left(\sum_{i=1}^{n}\left[\left(\frac{1}{1+r}\right)^i (C_i - M_i)\right]\right)$$

FINDING THE FIRST ORDER CONDITION FOR A PARTICULAR i:

$$\frac{\partial L}{\partial C_i} = \left(\frac{1}{1+\rho}\right)^i \frac{du}{dC_i} - \lambda\left(\frac{1}{1+r}\right)^i = 0$$

Therefore we get, for any i, j $\left(\frac{1+r}{1+\rho}\right)^i\left(\frac{du_i}{dC_i}\right) = \left(\frac{1+r}{1+\rho}\right)^j\left(\frac{du_j}{dC_j}\right)$

IF WE ASSUME LOGARITHMIC UTILITY SO THAT $u(C_i) = \ln(C_i)$ then $\frac{du_i}{dC_i} = \frac{1}{C_i}$, SO THE ABOVE CONDITION BECOMES:

$$\left(\frac{1}{C_i}\right)\left(\frac{1+r}{1+\rho}\right)^i = \left(\frac{1}{C_j}\right)\left(\frac{1+r}{1+\rho}\right)^j$$

THIS IS THE <u>RAMSEY</u> MODEL

IF WE THEN REARRANGE, WE GET:

$$C_i = \left(\frac{1+r}{1+p}\right)^{i-j} C_j$$

IF WE LET $i = j+1$ AND THEN TAKE NATURAL LOGS OF BOTH SIDES, WE GET:

$$\ln(C_{j+1}) = \ln(1+r) - \ln(1+p) + \ln(C_j)$$

Now for small r, p $\ln(1+r) \simeq r$ AND $\ln(1+p) \simeq p$,

SO WE GET $\ln(C_{j+1}) - \ln(C_j) \simeq r - p$

THE LEFT HAND SIDE IS AN APPROXIMATION FOR $\frac{\Delta C}{C}$ (THE PROPORTIONAL CHANGE IN C),

SO THIS BECOMES

$$\frac{\Delta C}{C} \simeq r - p$$

ALSO $r = A\left(\frac{dF}{dK}\right) - \delta$

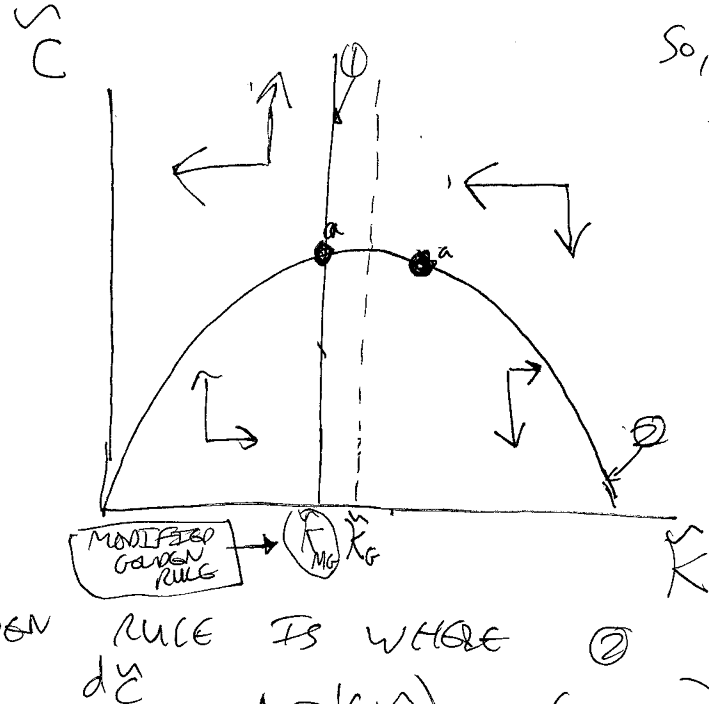
$\Rightarrow r = A\left(\frac{dF}{dK}\right)(L)\left(\frac{1}{L}\right) - \delta$

$\Rightarrow r = Af'(\hat{K}) - \delta$

SO, THIS GIVES US OUR SECOND EQUATION TO DESCRIBE THE DYNAMICS OF THE MODEL

$$\frac{\dot{\hat{C}}}{\hat{C}} = Af'(\hat{K}) - \delta - p \quad (1)$$

(AND, FROM BEFORE: $\dot{\hat{K}} = Af(\hat{K}) - \hat{C} - \delta\hat{K} - n\hat{K}$ (2)



SO, FOR $\hat{C}$ CONSTANT, WE NEED $\boxed{Af'(\hat{K}) = \delta + p}$ (1)

FOR $\hat{K}$ CONSTANT, WE NEED $\boxed{\hat{C} = Af(\hat{K}) - (\delta + n)\hat{K}}$ (2)

SO MODIFIED GOLDEN RULE IS WHERE $Af'(\hat{K}) = \delta + p$

MODEL ONLY WORKS IF $p > n$

GOLDEN RULE IS WHERE (2) REACHES MAXIMUM POINT:

$$\frac{d\hat{C}}{d\hat{K}} = Af'(\hat{K}) - (n+\delta) = 0$$

$$\Rightarrow Af'(\hat{K}) = n + \delta$$

EXOGENOUS AND ENDOGENOUS GROWTH

CONSIDER A PRODUCTION FUNCTION WITH LABOUR-AUGMENTING TECHNOLOGY:

$$Y(K,L) = F(K, AL)$$

FOR SIMPLICITY, WE SHALL USE A COBB-DOUGLASS PRODUCTION FUNCTION:

$$Y(K,L) = K^{\alpha}(AL)^{1-\alpha}$$

WE CAN DECOMPOSE INTO THE GROWTH CONTRIBUTED BY DIFFERENT COMPONENTS BY TOTALLY DIFFERENTIATING WITH RESPECT TO TIME:

$$g_Y = \frac{\dot{Y}}{Y} = \frac{\dot{K}\alpha K^{\alpha-1}(AL)^{1-\alpha} + (\dot{A}L + \dot{L}A)(1-\alpha)K^{\alpha}(AL)^{-\alpha}}{K^{\alpha}(AL)^{1-\alpha}}$$

$$\Rightarrow g_Y = \alpha\frac{\dot{K}}{K} + (1-\alpha)\left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right)$$

$$\Rightarrow g_Y = \alpha g_K + (1-\alpha)(g_A + n)$$ (WHERE n IS POPULATION GROWTH)

NOW, USING THE CAPITAL ACCUMULATION IDENTITY:

$$g_K = \left(\frac{\dot{K}}{K}\right) = \frac{sY - \delta K}{K} = s\left(\frac{Y}{K}\right) - \delta$$

NOW CONSIDER $\check{y} = \frac{Y}{L}$. THUS: $\dot{\check{y}} = \frac{L\dot{Y} - Y\dot{L}}{L^2} = \frac{LYg_Y - Y\dot{L}}{L^2}$

$$\Rightarrow \dot{\check{y}} = \check{y}(g_Y - n) \Rightarrow \frac{\dot{\check{y}}}{\check{y}} = \alpha\left(\frac{sY}{K} - \delta - n\right) + (1-\alpha)g_A$$

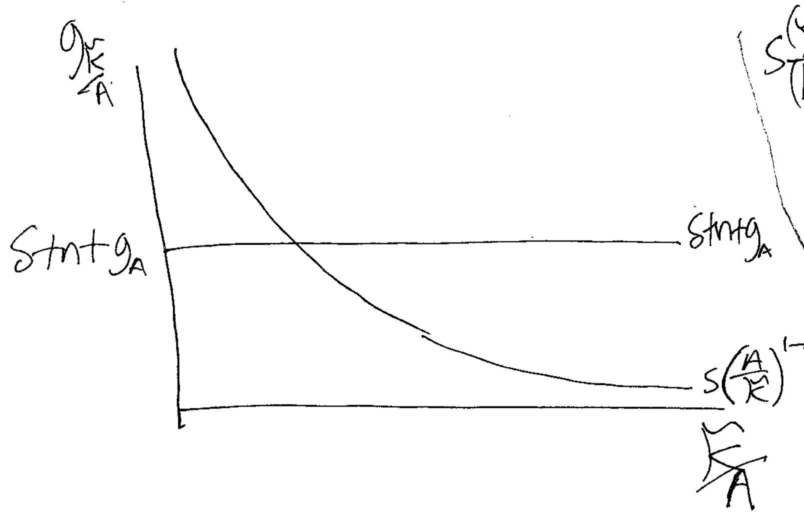
$$\Rightarrow \frac{\dot{\check{y}}}{\check{y}} = \alpha\left(\frac{s\check{y}}{\check{k}} - \delta - n\right) + (1-\alpha)g_A = g_{\check{y}}$$

MEANWHILE $\hat{K} = \frac{K}{L}$ AND SO $\dot{\hat{K}} = \frac{L\dot{K} - K\dot{L}}{L^2}$

$$\Rightarrow \dot{\hat{K}} = \frac{L g_K K - K L n}{L^2} \Rightarrow \dot{\hat{K}} = g_K \hat{K} - \hat{K} n$$

$$\Rightarrow \frac{\dot{\hat{K}}}{\hat{K}} = \frac{s\hat{Y}}{\hat{K}} - \delta - n = g_{\hat{K}}$$

SO, WE CAN SEE THAT, IF $g_{\hat{K}} < g_A$ THEN $g_{\hat{Y}} > g_{\hat{K}}$ AND SO $g_{\hat{K}}$ WILL BE INCREASING AND IF $g_{\hat{K}} > g_A$ THEN $g_{\hat{Y}} < g_{\hat{K}}$ AND SO $g_{\hat{K}}$ WILL BE DECREASING. THUS $g_{\hat{K}}$ AND $g_{\hat{Y}}$ WILL CONVERGE TO g_A. IN STEADY STATE



$$s\frac{(\hat{Y}/A)}{(\hat{K}/A)} - \delta - n = g_A$$

$$s\frac{(\hat{Y}/A)}{(\hat{K}/A)} = \delta + n + g_A$$

$$\hat{Y}/A = \frac{Y}{LA} = \frac{K^\alpha (AL)^{1-\alpha}}{LA}$$

$$\hat{Y}/A = A^{-\alpha}\left(\frac{K}{L}\right)^\alpha$$

$$\hat{Y}/A = \left(\frac{\hat{K}}{A}\right)^\alpha$$

THESE MECHANICS OF CONVERGENCE OCCUR AS LONG AS g_A IS CONSTANT ALONG A STEADY STATE/BALANCED GROWTH PATH.

$$\Rightarrow s\frac{\hat{Y}}{\hat{K}} = \frac{sA^{1-\alpha}}{\hat{K}^{1-\alpha}} = s\left(\frac{A}{\hat{K}}\right)^{1-\alpha}$$

HOWEVER, IN THE ROMER MODEL $g_Y = cL_R$ WHERE L_R IS THE NUMBER OF PEOPLE ENGAGED IN R+D. HENCE AS $\hat{K}$ INCREASES, IF THIS IS DUE TO INCREASES IN BOTH K AND L (WITH K INCREASING PROPORTIONALLY MORE), THEN g_A INCREASES AND SO $\hat{Y}$ CAN CONTINUE TO INCREASE EXPLOSIVELY WITHOUT CONVERGENCE. IN THE JONES MODEL, ON ~~THE~~ OTHER HAND, IT CAN BE SHOWN THAT $g_A = \frac{n}{1-\eta} = g_{\hat{K}} = g_{\hat{Y}}$

JONES AND ROMER ENDOGENOUS GROWTH MODELS

BOTH USE SIMILAR FORMULATION FOR TECHNOLOGY DEVELOPMENT AND PRODUCTION FUNCTION:

$$Y = K^{\alpha}(AL_Y)^{1-\alpha} \qquad \dot{A} = cA^{\gamma}L_R \qquad L = \underbrace{L_Y}_{\text{"NORMAL" WORKERS}} + \underbrace{L_R}_{\text{R+D WORKERS (RESEARCH AND DEVELOPMENT)}}$$

$$\Rightarrow \quad g_A = \frac{\dot{A}}{A} = cA^{\gamma-1}L_R$$

<u>ROMER MODEL</u>: $\gamma = 1 \Rightarrow g_A = cL_R$

IN THIS MODEL, ASSIGNING MORE LABOUR TO R+D WILL PERMANENTLY RAISE THE GROWTH RATE OF PER CAPITA GDP.

<u>JONES MODEL</u>: $0<\gamma<1 \Rightarrow g_A = c\frac{L_R}{A^{1-\gamma}}$

ALONG A BALANCED GROWTH PATH, g_A IS CONSTANT

$$\Rightarrow \frac{\frac{d}{dt}(L_R)}{L_R} = \frac{\frac{d}{dt}\left(A^{1-\gamma}\right)}{A^{1-\gamma}} \Rightarrow \frac{\dot{L}_R}{L_R} = \frac{(1-\gamma)\dot{A}A^{-\gamma}}{A^{1-\gamma}}$$

$$\Rightarrow \quad n = (1-\gamma)g_A$$

(ASSUMING A FIXED PROPORTION OF WORKERS IN R+D SO THAT $\frac{\dot{L}_R}{L_R} = n$)

$$\Rightarrow \quad g_A = \frac{n}{1-\gamma}$$

THE BALANCED GROWTH PATH NOW INVOLVES A CONSTANT RATE OF GROWTH ~~OF~~ PER CAPITA GDP WHICH IS INCREASING IN POPULATION GROWTH.

THIS IS ARGUABLY MORE REALISTIC SINCE IT DOES NOT ALLOW EXPLOSIVE GROWTH AS POPULATION INCREASES.

TECHNOLOGY TRANSFER

$$g_{A_i} = \phi_i \left(\frac{A_{us} - A_i}{A_i} \right)$$

IN STEADY STATE g_{A_i} IS CONSTANT OVER TIME

$$g_{A_i} = \phi \left(\frac{A_{us}}{A_i} - 1 \right)$$

SO, TIME DERIVATIVE OF $\frac{A_{us}}{A_i}$ MUST BE ZERO

$$\Rightarrow \frac{A_i \dot{A}_{us} - A_{us} \dot{A}_i}{(A_i)^2} = 0$$

$$\Rightarrow \left(\frac{\dot{A}_{us}}{A_{us}} \right) \left(\frac{A_{us}}{A_i} \right) - \left(\frac{A_{us}}{A_i} \right) \left(\frac{\dot{A}_i}{A_i} \right) = 0$$

$$\Rightarrow \left(\frac{A_{us}}{A_i} \right) \left(g_{A_{us}} - g_{A_i} \right) = 0$$

$$\Rightarrow g_{A_{us}} = g_{A_i}$$

$$\Rightarrow g_{A_{us}} = \phi_i \left(\frac{A_{us} - A_i}{A_i} \right)$$

$$A_i \left(g_{A_{us}} + \phi_i \right) = \phi_i A_{us}$$

$$\frac{A_i}{A_{us}} = \frac{\phi_i}{g_{A_{us}} + \phi_i}$$

THIS MEANS THAT IN A STEADY STATE TECHNOLOGY IN COUNTRY i GROWS AT THE SAME RATE AS THE FRONTIER, BUT REMAINS PERMANENTLY "STUCK" BEHIND