# Week 6 - Monopoly

November 26, 2007

## 1

Suppose that a monopolist is faced by a linear demand curve $X_D = A - bP_X$ and has a constant marginal cost of C.

(a) To find the monopolists profit-maximizing price and output, we first form the profit function $\pi = X_D P_x - CX_D$. Since price and quantity change together, in order to differentiate this to find the maximum we must either substitute in the inverse demand curve to eliminate $P_X$ and then differentiate with respect to $X_D$, or substitute in the demand curve to eliminate $X_D$ and differentiate with respect to $P_X$. Both methods, if applied correctly, should give the same answer for the profit-maximizing price and output. We will here eliminate $X_D$, since we will be differentiating with respect to quantity shortly when we find $MR_X$, and differentiating with respect to price is good practice as it is necessary when you have a monopolist who operates in two markets and must set the same price in both (it is necessary to analyse this situation when considering the effect of third degree price discrimination, as you will see in the vacation work). Substituting the expression for the demand curve into the profit function gives us: $\pi = (A - bP_X)P_x - C(A - bP_X) = AP_X - bP_X{}^2 - AC + bP_X C$. Differentiating this yields: $\frac{\partial \pi}{\partial p_X} = A - 2bP_X + bC$. Setting this equal to 0 and rearranging yields $P_X^* = \frac{A+bC}{2b} = \frac{A}{2b} + \frac{C}{2}$ for the profit-maximizing price. Plugging this into the expression for the demand curve yields: $X_D = A - b\left(\frac{A}{2b} + \frac{C}{2}\right) = \frac{A}{2} - \frac{bC}{2}$. These results have a straightforward intuitive interpretation. With a linear demand curve and constant marginal costs, the marginal revenue curve comes down at precisely twice the slope of the demand curve and thus hits the horizontal marginal cost curve half way between the y-axis and the point where the demand curve hits the horizontal marginal cost curve. This implies that the monopolist jacks the price up half way from the marginal cost to the maximum reservation price where the demand curve hits the y-axis at $\frac{A}{b}$, and produces half the competitive output $A - bC$.

(b) The inverse demand curve is given by the expression $\frac{A}{b} - \frac{X_D}{b}$. To find marginal revenue, we generally differentiate with respect to quantity, since it tells us how much revenue increases by when output increases by 1 unit. In this case revenue $R_X = X_D P_X = X_D \left(\frac{A}{b} - \frac{X_D}{b}\right) = \frac{A}{b} X_D - \frac{1}{b} {X_D}^2$. Differentiating this yields $MR_X = \frac{dR_X}{dX_D} = \frac{A}{b} - \frac{2}{b} X_D$. This establishes algebraically the result that the MR curve has the same y-intercept as the demand curve but comes down at twice the slope. MR is equal to 0 when $X_D = \frac{A}{2}$

(c) Price elasticity of demand is given by $\epsilon_X = \frac{dX_D}{dP_X} \frac{P_X}{X_D}$. In this case, this simplifies to give $\epsilon_X = -b \frac{\frac{A}{b} - \frac{X_D}{b}}{X_D} = \left(1 - \frac{A}{X_D}\right)$. Now, plugging in $X_D = \frac{A}{2}$ can be seen to give $\epsilon_X = -1$. This shows that the point of unit elasticity is reached half way along a linear demand curve. This is where the revenue rectangle under the demand curve is maximized, because any further increase in quantity causes a proportionally greater reduction in price. At the maximum itself, a marginal change in price causes a precisely proportional marginal change in output.

(d) For any output greater than the point of unit elasticity ($\frac{A}{2}$ in the above example) elasticity is less than 1 in absolute value and marginal revenue is *negative*. This means that the monopolist can increase its revenue by *decreasing* its quantity sold. Not only that, this would also decrease its costs. Hence the monopolist can never be maximizing its profits on the inelastic part of the demand curve (another way to see this is that since MC is always positive, it cannot possibly cross the MR curve where that has gone negative).

(e) (f) A competitive firm would have to charge $P_{XC} = C$ and sell $X_{DC} = A - bC$, since otherwise it would be possible for other firms to enter the market and undercut its price at a profit. This would also be the Pareto-efficient price, since only at this price is there no deadweight loss, and the presence of a deadweight loss always indicates that there can potentially be a reallocation of resources that, with suitable side payments, makes everybody better off.

(g) The deadweight loss is the sum (integral) of the difference between reservation prices and marginal cost over the output distortion. Since the marginal cost is constant, in this case it is equal to the area of the triangle with base equal to the output distortion $(\frac{A}{2} - \frac{bC}{2})$ and height equal to the excess of the monopoly price over $C$ $(\frac{A}{2b} + \frac{C}{2} - C = \frac{A}{2b} - \frac{C}{2})$. Hence the area of the triangle simplifies to give $\frac{1}{2b}\left(\frac{A}{2} - \frac{bC}{2}\right)^2$. So, the deadweight loss is proportional to the square of the output distortion (this will be approximately true even in the general case with non-linear demand and supply curves).

# 2

See week 7 notes.

# 3

Suppose that a monopolist is faced by a constant elasticity demand curve $X_D = AP_X^{-\gamma}$, where $\gamma > 1$, and has a constant marginal cost of production $C$.

(a) The price elasticity of demand is given by $\epsilon_X = \frac{dX_D}{dP_X}\frac{P_X}{X_D}$. In this case, $\frac{dX_D}{dP_X} = -\gamma AP^{-\gamma-1}$. Substituting this expression, along with that for the demand curve, into the elasticity expression and simplifying yields a constant elasticity of $-\gamma$. Therefore, if $\gamma$ were less than 1, the demand function would be inelastic at every point. This would result in the monopolist driving the price up to infinity and selling an infinitely small amount for an infinitely high revenue. This clearly would not make sense.

(b) We will now derive the monopoly mark-up above marginal cost. The monopolist's profit function will be given by $\pi_X = P_X X_D - CX_D$. Substituting in the expression for the demand curve yields: $\pi_X = P_X\left(AP_X^{-\gamma}\right) - C\left(AP_X^{-\gamma}\right) = AP_X^{1-\gamma} - ACP_X^{-\gamma}$. Diffentiating with respect to price and setting equal to 0 to find the maximum yields: $\frac{d\pi}{dP_X} = (1-\gamma)AP_X^{-\gamma} + \gamma ACP_X^{-\gamma-1} = 0$. Factorizing this so that we can find the roots gives us $AP_X^{-\gamma}\left((1-\gamma) + \gamma CP_X^{-1}\right) = 0$. Only the part in brackets can equal 0, and this occurs when $(1-\gamma) + \gamma\frac{C}{P_X} = 0$. Rearranging yields: $\frac{P_X^*}{C} = -\frac{\gamma}{1-\gamma} = \frac{1-\gamma}{1-\gamma} + \frac{-\gamma-(1-\gamma)}{1-\gamma} = 1 - \frac{1}{1-\gamma}$. Now, since $-\gamma = \epsilon_D$, we can see that $\frac{P_X^*}{C} = 1 - \frac{1}{1+\epsilon_D}$. This is the mark-up. As $\epsilon_D \longrightarrow -\infty$, demand becomes infinitely elastic and the mark-up goes to 1. As $\epsilon_D \longrightarrow -1$ (always, remember, staying less than, i.e. more negative than, -1), the mark-up goes to infinity. The intuition for these results is that if demand is very elastic, the monopolist has very little market power because it rapidly loses sales if it puts the price up. If, on the other hand, demand is very inelastic, the monopolist is able to increase the price very far above marginal cost because consumers will still buy the good. Thus, monopolies over goods with very inelastic demand (e.g. food, oil) are particularly harmful.