Syntax, Truth, Paradox:

A map through the land of dragons

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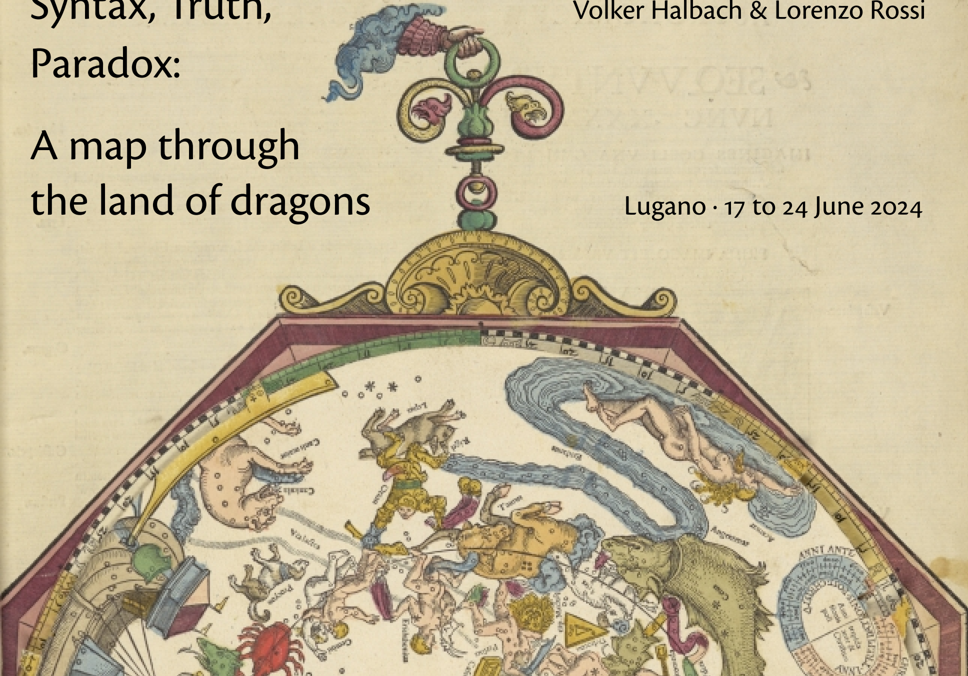


Prelude: The paradoxes in philosophy

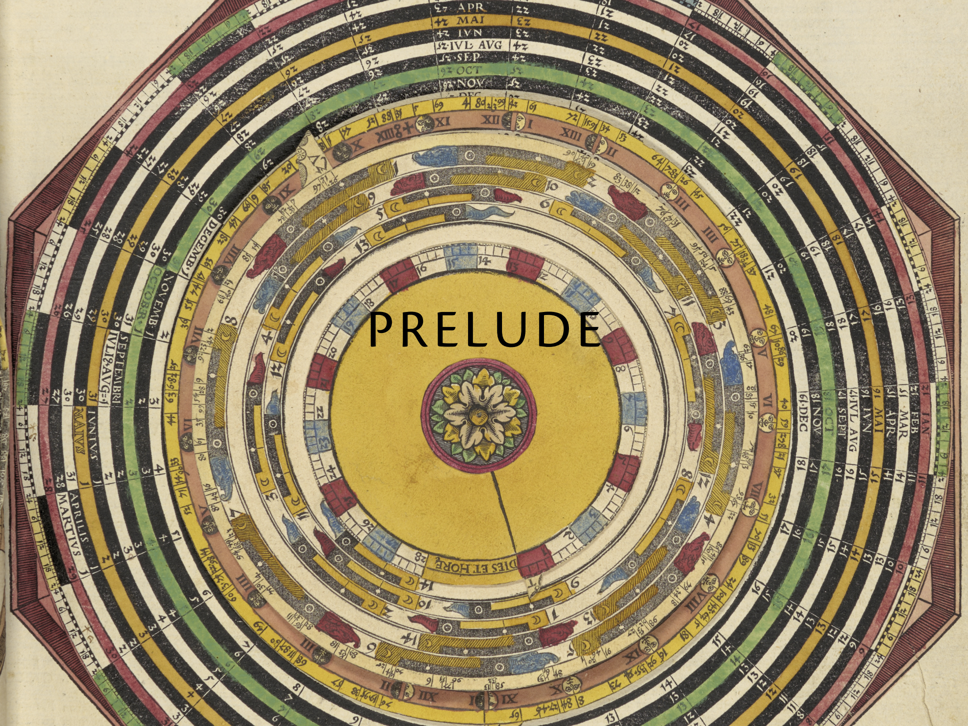
Day 1: A theory of expressions

Day 2: The paradoxes

Day 3: Possible-worlds analysis of the paradoxes

Day 4: Truth!

PRELUDE



Modal Notions and Paradox

modal notions

is necessary, is true, is a priori (true), is analytic, is obligatory, S knows, is knowable, is verifiable …

They are often combined with *that*-clauses, the *dictum*.

They apply to sentences, beliefs, propositions, states of affairs, and the like.

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If a belief is not true, it cannot be known. ('Knowledge implies truth.')

All true propositions (or sentences) can be verified.

There are sentences that a necessarily true, but cannot be known a priori.

There are true sentences that cannot be proved in Peano arithmetic.

If the premisses of a logically valid argument are true, its conclusion is also true.

Moral judgements are neither true nor false.

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(D) (D) is not demonstrable.

The point of labelling the sentence with '(D)' is that we have the following identity:

(D) = '(D) is not demonstrable'.

Demonstrability can be understood in different ways. It applies on straightforward understandings to sentences.

1. Assume (D) is demonstrable.
2. Then '(D) is not demonstrable' is demonstrable (by the identity above).
3. Therefore, (D) is not demonstrable.
4. Hence, (D) is not demonstrable (because the first line and the previous line, derived from it, contradict each other, and thus the assumption in the first line is refuted).
5. That is, we have just verified (D) (because the preceding sentence is just (D)).

Modal notions are threatened by paradox.

Truth is only a special case.

What is going wrong?

1. We cannot label a sentence and use that very label in the sentence.
2. We have to change our logic. Something is wrong with classical logic.
3. Demonstrability is a dodgy notion and should be eliminated or replaced, e.g., with provability in a specific formal system.

Self-reference

'We cannot label a sentence and use that very label in the sentence.'

Of course we can. I just did. But one might claim that such labels should be ruled out.

The red sentences on this page is not demonstrable.

This very sentence is not demonstrabe.

I am not demonstrable.

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Self-reference

Quine (1976): The quotation of an expression is the expression enclosed in quotation marks.

> *'preceded by its own quotation is not demonstrable' preceded by its own quotation is not demonstrable.*

We will study purely syntactic ways of obtaining this effect.

Indirect self-reference can also cause problems:

postcard paradox

We have a postcard: One side says: ‘The sentence of the other page is true.’, he other: ‘The sentence on the other page is not true.’ Nothing else is written on the postcard.

1. Sentence 2 is true.
2. Sentence 1 is not true.

Technical Preliminaries

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Function symbols?

Models?

Calculi?

Operators and predicates

Modal logic

Necessity is one of the paradox-prone modal notions.

'is necessary'

'expresses a necessary proposition'

Logicians have formalized necessity (especially metaphysical necessity) as $\Box$ in modal logic.

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operators and predicate analysis

It is necessary that water is H_2O

can be parsed in at least two different ways. According to the first option, 'it is necessary that' is combined with the sentence 'Water is H_2O':

$$\underbrace{\textit{It is necessary that}}_{\textit{operator, adverb}}\ \underbrace{\textit{water is } \mathrm{H_2O}.}_{\textit{sentence}}$$

In this sentence the phrase 'it is necessary that' serves the same purpose as the adverb 'necessarily' in the following sentence:

$$\underbrace{\textit{Necessarily}}_{\textit{operator}}\ \underbrace{\textit{water is } \mathrm{H_2O}.}_{\textit{sentence}}$$

$$\underbrace{\textit{It is necessary}}_{\textit{predicate}}\ \underbrace{\textit{that water is } \mathrm{H_2O}}_{\textit{singular term}}.$$

$$\underbrace{\textit{That water is } \mathrm{H_2O}}_{\textit{singular term}}\ \underbrace{\textit{is necessary.}}_{\textit{predicate}}$$

$$\underbrace{\textit{The proposition that water is } \mathrm{H_2O}}_{\textit{singular term}}\ \underbrace{\textit{is necessary.}}_{\textit{predicate}}$$

But compare:

Donald fears that there will be more indictments.

and

Donald fears the proposition that there will be more indictments.

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Carnap and Quine used predicates and stayed in first-order logic.

The analysis can be applied to other modal notions. On the predicate account, we need ot decide to which kind objects the notion apply (fine- or coarse grained propositions, sentences, beliefs etc.)

In a formal language a (unary sentential) operator has the same grammar as $\neg$.: You attach it to a formula φ and obtain a new formula.

I write $\Box$ for necessity and $\Diamond$ for possibility *operators*. Apologies!

So, if φ is a formula, $\Box\varphi$ and $\Diamond\varphi$ are formulæ. All three formulæ contain the same free variables.

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In English we have adverbs and predicates. We also have verb modification tenses, conjunctive, potentialis etc.

Problems of the predicate approach

(i) The usual possible-worlds semantics is not applicable.

(ii) Paradoxes arise.With an adverb the paradoxes cannot be formed.

(iii) No ontology of the objects to which the modal notions apply is required, at least not in the object language. Cf. 'There are synthetic truths *a priori*' and Kripke's (1979) Pierre.

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Problems of the operator approach

'There are synthetic truths *a priori*' cannot be formalized.

Generally, quantified claims cannot be formalized – unless additional resources are introduced.

My favourite example of confusion about operators and predicates

Gettier (1963) (and copied in half of all epistemology books):

S knows that P IFF (i) P is true,

(ii) S believes that P, and

(iii) S is justified in believing that P.

Expressive strength

If we have a term forming device, every sentence with an operator can be expressed with a predicate.

Term forming devices:

Snow is white $\Longrightarrow$ ‘Snow is white’

Snow is white $\Longrightarrow$ the proposition that snow is white

Reduction of operators to predicates

Replace

Necessarily, snow is white

with 'Snow is white is necessary' or 'The proposition that snow is white is necessary.'

Formally,

Replace $\underline{\Box}\varphi$ with $\Box\overline{\varphi}$.

Here $\Box$ is now a predicate, while $\underline{\Box}$ is the operator. $\overline{\varphi}$ is a quotation name for φ.

For a reduction in the other direction we need funny quantifiers or other additional devices (e.g. truth). See (Kripke 1975, Stern 2016).

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De re modality

In standard operator modal logic we distinguish:

$$\forall x(Px \rightarrow \underline{\Box} Qx) \text{ and}$$
$$\underline{\Box} \forall x(Px \rightarrow Qx)$$

In the first formula $\underline{\Box}$ can be read as metaphysical necessity, belief, knowledge, but not as analyticity or provability.

How can *de re* modality formalized with a predicate for the modality?

We could employ a binary predicate 'is necessary for'; but we may need two free variables, which would require a ternary predicate, and so on.

Tarski (1935) solved the problem for truth with a binary satisfaction predicate applying to formulæ and variable assignments.

The same can be done for other modalities. This further increases the expressive power of the predicate approach compared to the operator approach, but it also requires resources for forming variable assignments.

See (Halbach 2021) and the literature on 'disentangled' syntax.

Paradoxes over Logic

Famously, Ramsey (1926) introduced the distinction between semantic and set-theoretic paradoxes. But it is not so clear that they are deeply distinct.

$$\exists x \forall y\,(y \in x \leftrightarrow \varphi(y))$$

$$\forall y\,(y \in \overline{\varphi} \leftrightarrow \varphi(y))$$

$$\forall y\,(\mathsf{Sat}\,\overline{\varphi}\,y \leftrightarrow \varphi(y))$$

x is not free in φ, and $\overline{\varphi}$ doesn't occur in φ.

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