Syntax, Truth, Paradox:

A map through the land of dragons

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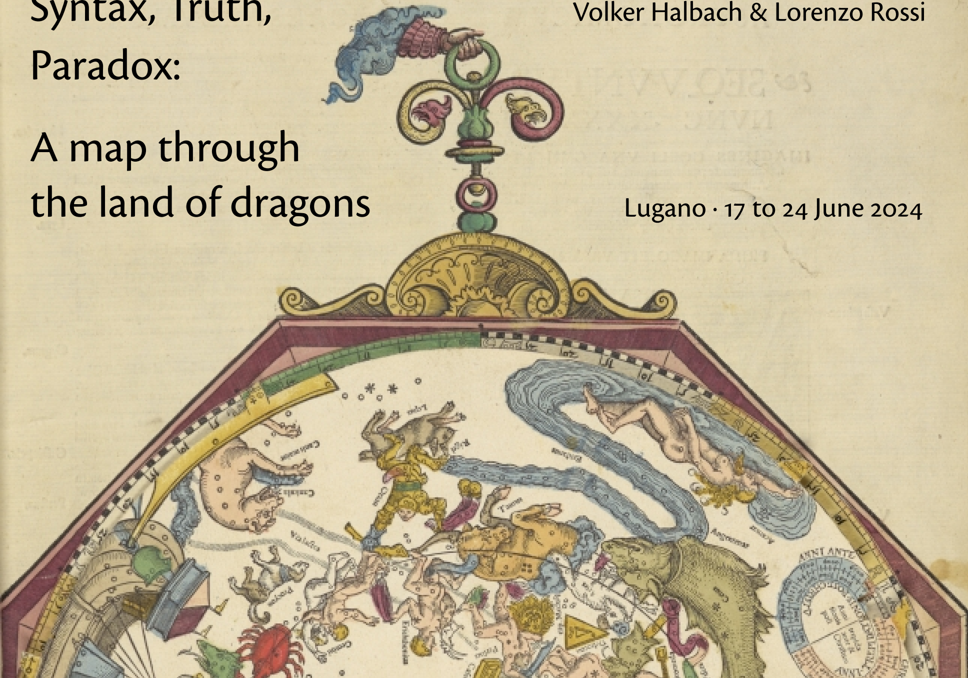

Prelude: The paradoxes in philosophy

Day 1: A theory of expressions

Day 2: The paradoxes

Day 3: Possible-worlds analysis of the paradoxes

Day 4: Truth!

POSSIBLE WORLDS

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Why possible worlds semantics for predicate?

Possible worlds for sentential operators

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Negative results

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We obtain an analysis of the paradoxes.

Paradoxes can be 'visualized'.

It shows that the predicate approach does not not force us to abandon insights obtained via pws for sentential operators.

It allows one to find a common cause for many paradoxes.

It establishes a bridge to modal metaphysics.

It sheds a light on classical questions such as ante rem/in rebus conceptions of properties.

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Possible worlds for sentential operators

The language of (sententential operator) modal logic treats the modal operator just like $\neg$.

If φ is a formula, so is $\underline{\Box}\varphi$, whereas for the modal *predicate* $\Box$ we can write at best $\Box\overline{\varphi}$ or $\Box x$.

Consequently, we can define the logical complexity of a formula in the lanugage of (sententential operator) modal logic ion the usual way.

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A model for a language with a modal operator usually specifies a non-empty set W of worlds, an accessibility relation R on W, and an interpretation V that assign an interpretation to the non-logical vocabulary. For a standard first-order language, V gives applied to a world $w \in W$ a domain and to a world and a n-ary predicate P symbol an n-ary relation, and so on.

The truth of $\Box A$ at a world is then defined by induction on the complexity of A simultaneously for all worlds.

This is possible because formulæ in modal logic are wellfounded.

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Fundamentals

For pws semantics I make further assumption about the language $\mathcal{L}$.

$\mathcal{L}$ is the minimal language plus a single sentence symbol p. This the lanugage $\mathcal{L}_p$.

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Of course, we would like to have more ‘contingent’ vocabulary. Here I keep things simple.

The standard model $\mathbb{E}$ of $\mathcal{L}_{\text{qc}}$ has as its domain the set of all expressions of the full language $\mathcal{L}$ and interprets the vocabulary such as the quotation constants and $\frown$ in the expected way.

A standard model for $\mathcal{L}_p$ is of the form $\langle \mathbb{E}, V, S \rangle$, where V assigns a truth value *true* or *false* to p and S is the extension of $\Box$, that is, we have the following:

$$\langle \mathbb{E}, V, S \rangle \models \Box \overline{e} \quad \text{iff} \quad e \in S.$$

The notion of a frame is exactly the same as in operator modal logic:

Definition

A *frame* is an ordered pair $\langle W, R \rangle$ where W is non-empty and R is a binary relation on W.

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The notion of a frame is exactly the same as in operator modal logic:

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A *frame* is an ordered pair $\langle W, R \rangle$ where W is non-empty and R is a binary relation on W.

Definition

A *PW-model* is a quadruple $\langle W, R, V, B\rangle$ such that $\langle W, R\rangle$ is a frame, V is a valuation for $\langle W, R\rangle$, and B is a $\Box$-interpretation for $\langle W, R\rangle$ satisfying the following condition, where $\mathcal{L}_p$ is the set of all $\mathcal{L}_p$-sentences:

$$B(w) = \left\{ \varphi \in \mathcal{L}_p \colon \text{ for all } u \in W\text{: if } wRu \text{ then } \left\langle \mathbb{E}, V(u), B(u) \right\rangle \vDash \varphi \right\}.$$

$\langle \mathbb{E}, V(u), B(u) \rangle \vDash \varphi$ means that the sentence φ is true in the standard model $\langle \mathbb{E}, V(u), B(u) \rangle$ in the usual sense of first-order predicate logic; and the expression $\langle \mathbb{E}, V(u), B(u) \rangle \vDash \varphi$ can be read as 'φ is true at world u in the PW-model $\langle W, R, V, B \rangle$'.

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Lemma (normality)

Suppose $\langle W, R, V, B\rangle$ *is a* PW*-model,* $w \in W$*, and* φ*,* ψ *sentences of* $\mathcal{L}_P$*. Then the following hold:*

(i) *If* $\left\langle \mathbb{E}, V(u), B(u) \right\rangle \vDash \varphi$ *for all* $u \in W$*, then* $\left\langle \mathbb{E}, V(w), B(w) \right\rangle \vDash \Box\overline{\varphi}$*.*

(ii) $\left\langle \mathbb{E}, V(w), B(w) \right\rangle \vDash \Box\overline{\varphi \to \psi} \to (\Box\overline{\varphi} \to \Box\overline{\psi})$.

Lemma

(i) *If a frame* $\langle W, R\rangle$ *is transitive and* $\langle W, R, V, B\rangle$ *a* PW*-model on that frame, we have for all sentences* φ *in* $\mathcal{L}_p$ *and worlds* $w \in W$*:*

$$\left\langle \mathbb{E}, V(w), B(w)\right\rangle \vDash \Box\overline{\varphi} \rightarrow \Box\overline{\Box\overline{\varphi}}.$$

(ii) *If a frame* $\langle W, R\rangle$ *is reflexive and* $\langle W, R, V, B\rangle$ *a* PW*-model on that frame, we have for all sentences* φ *in* $\mathcal{L}_p$ *and worlds* $w \in W$*:*

$$\left\langle \mathbb{E}, V(w), B(w)\right\rangle \vDash \Box\overline{\varphi} \rightarrow \varphi.$$

Definition

A frame $\langle W, R\rangle$ *admits a* PW*-model on every valuation* iff for every valuation V on $\langle W, R\rangle$ there is a B such that $\langle W, R, V, B\rangle$ is a PW-model. A frame *admits a* PW*-model* iff the frame admits a PW-model on some valuation, that is, iff there is a valuation V and a $\Box$-interpretation B such that $\langle W, R, V, B\rangle$ is a PW-model.

strong characterizaiton problem

Which frames admit a PW-model on every valuation?

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Which frames admit a PW-model on every valuation?

Negative results

There is only one world, say, w, and this world can see itself. Thus we have $W_1 = \{w\}$ and $R_1 = \{\langle w, w\rangle\}$.

Theorem (liar paradox)

The above frame $\langle W_1, R_1\rangle$ does not admit a valuation.

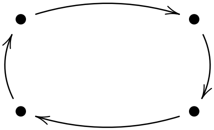
Example (Montague)

If $\langle W, R \rangle$ admits a valuation, then $\langle W, R \rangle$ is not reflexive.

Example

The frame ‘two worlds see each another’ displayed above does not admit a valuation.

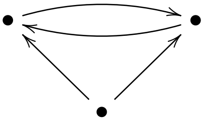
The following frame does not admit a PW-model.



Example

The frame ‘one world sees itself and one other world’ does not admit a valuation.

For the proof the fixed point $\gamma \leftrightarrow (\Box\overline{\gamma} \rightarrow \Box\overline{\neg\gamma})$ can be employed.



Example

The above frame ‘one world sees two worlds that see each another’ does not admit a valuation.

One can show this by using the fixed point $\gamma \leftrightarrow \neg\Box\overline{\Box\overline{\gamma}} \wedge \neg\Box\overline{\gamma}$

Example (McGee's paradox)

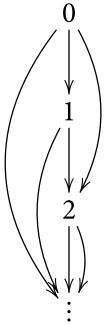
The frame $\langle \omega, \text{Pre} \rangle$ does not admit a PW-model. Here ω is the set of all natural numbers and Pre is the successor relation. Hence every world n sees $n + 1$ but no other world.

The frame $\langle \omega, \text{Pre} \rangle$ can be displayed by the following diagram:

Theorem (Yablo–Visser paradox)

The frame $\langle \omega, < \rangle$ does not admit a PW*-model. Here $<$ is the usual 'smaller than' relation on the natural numbers:*

The frame $\langle \omega, < \rangle$ can be displayed by the following diagram:



Positive results

By Suc we denote the successor relation $\{\langle k, n\rangle : k = n + 1\}$ on the set ω of natural numbers.

EXAMPLE

The frame $\langle \omega, \mathrm{Suc}\rangle$ admits a PW-model on every valuation.

Obviously we can stop at any point.

What about the liar sentence?

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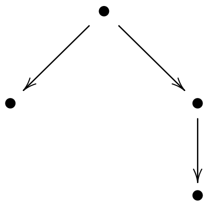
$$
\begin{gathered}
\left\langle \mathbb{E}, V(0), B(0) \right\rangle \vDash \neg\gamma, \\
\left\langle \mathbb{E}, V(1), B(1) \right\rangle \vDash \gamma, \\
\left\langle \mathbb{E}, V(2), B(2) \right\rangle \vDash \neg\gamma, \\
\left\langle \mathbb{E}, V(3), B(3) \right\rangle \vDash \gamma, \\
\vdots
\end{gathered}
$$



EXAMPLE

The frame above admits a PW-model on every valuation.

We can prove this for any tree of this kind.

Definition

A frame $\langle W, R\rangle$ is converse wellfounded (or Noetherian) iff for every non-empty $M \subseteq W$ there is a $w \in M$ that is R-maximal in M.

Lemma

Every converse wellfounded frame $\langle W, R\rangle$ admits a PW*-model on every valuation.*

Proof.

Define B by recursion on the rank of w in R^{-1}:

$$B(w) := \{\, \varphi \in \mathcal{L}_P \colon \forall u\,(wRu \to \langle \mathbb{E}, V(u), B(u)\rangle \vDash \varphi)\,\}$$

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The Strong Characterization Problem

Theorem (Strong Characterization theorem)

A frame admits a PW-model on every valuation iff it is converse wellfounded.

The lemma above yields the right-to-left direction.

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The accessibility relation need not be transitive. We can define its transitive closure:

$$\mathsf{E} \vdash \forall n \forall \mathsf{z} \forall \mathsf{y} \Big(\gamma(n, \mathsf{z}, \mathsf{y}) \ \leftrightarrow$$

$$\exists k \left(n = k^\frown \underline{1} \ \wedge \ \mathsf{z} = \overline{\forall \mathsf{z} (\gamma(\dot{k}, \mathsf{z}, \dot{\mathsf{y}}) \to \Box \mathsf{z})} \right) \vee \left(n = \underline{0} \wedge \mathsf{z} = \mathsf{y} \right) \Big)$$

$$\Box^0 \overline{\varphi} := \Box \overline{\varphi},$$

$$\Box^{n+1} \overline{\varphi} := \Box \overline{\Box^n \overline{\varphi}}$$

Lemma

For every PW*-model* $\langle W, R, V, B \rangle$*, every* $w \in W$*, and all* $n \in \mathbb{N}$*,*

$$\big\langle \mathbb{E}, V(w), B(w) \big\rangle \vDash \forall \mathsf{z} \big(\gamma(\underline{n}, \mathsf{z}, \overline{\varphi}) \to \Box \mathsf{z} \big) \leftrightarrow \Box^n \overline{\varphi}$$

Then we define $\Box^*$ as follows:

$$\Box^* \mathsf{y} \ := \ \forall n \forall \mathsf{z} \big(\gamma(n, \mathsf{z}, \mathsf{y}) \to \Box \mathsf{z} \big).$$

Lemma

For all φ *in* $\mathcal{L}_p^{S}$, *all* PW*-models* $\langle W, R, V, B\rangle$, *and all* $w \in W$,

$$\left\langle \mathbb{E}, V(w), B(w) \right\rangle \vDash \Box^{*}\overline{\varphi} \quad \textit{iff} \quad \textit{for all } v \textit{ with } wR^{*}v\textit{:}\ \left\langle \mathbb{E}, V(v), B(v) \right\rangle \vDash \varphi.$$

Lemma

For all $\mathcal{L}_P$-sentences φ and ψ and PW*-models $\langle W, R, V, B\rangle$ the following hold:*

(i) *If* $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \models \varphi$ *for all* $w \in W$, *then* $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \models \Box^* \overline{\varphi}$.

(ii) $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \models \Box^* \overline{\varphi \to \psi} \to (\Box^* \overline{\varphi} \to \Box^* \overline{\psi})$.

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(iv) $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \models \Box^* \overline{\Box^* \overline{\varphi} \to \varphi} \to \Box^* \overline{\varphi}$.

Lemma

The transitive closure R^* of the accessibility relation R of any frame that admits a PW-model on every valuation is converse well-founded.

Assume that $\langle W, R^* \rangle$ is converse ill-founded. Then there is a non-empty set $M \subseteq W$ without an R^*-maximal element. Define a valuation V as follows:

$$V(w)(p) := \begin{cases} \text{true}, & \text{if } w \notin M, \\ \text{false}, & \text{if } w \in M. \end{cases}$$

Lemma

A frame $\langle W, R \rangle$ is converse well-founded iff its transitive closure $\langle W, R^* \rangle$ is converse well-founded.

Theorem (Strong Characterization theorem)

A frame admits a PW-model on every valuation iff it is converse wellfounded.

DEEP INSIGHT

Löb's theorem is the mother of all paradoxes.

– at least in settings where we have pws.

sectionDe re modality