Syntax, Truth, Paradox:

A map through the land of dragons

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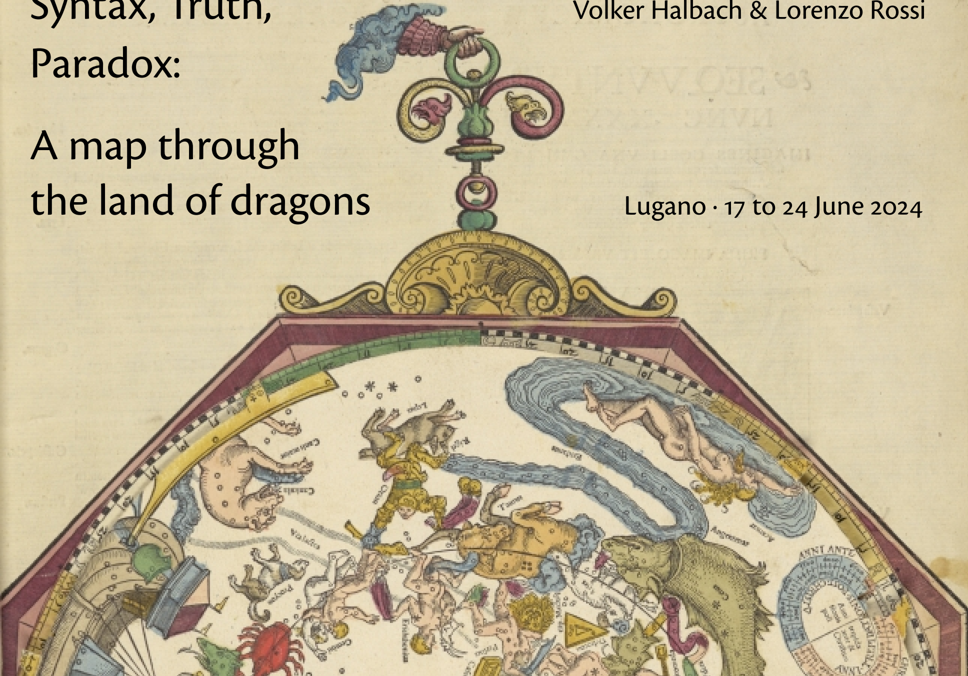

Prelude: The paradoxes in philosophy

Day 1: A theory of expressions

Day 2: The paradoxes

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Day 4: Truth!


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De re modality

An expressively strong syntax theory

The syntax theory E is weak (in a way I am going to explain).

This allowed me to prove *stronger* results about paradoxes.

Now we change perspective: I will formulate theories of truth and prove their consistency.

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DEFINITION

All instances of the following schemas are axioms of E:

A1 $\overline{a} ^\frown \overline{b} = \overline{ab}$, where a and b are arbitrary strings of symbols

A2 $\mathsf{q}(\overline{a}) = \overline{\overline{a}}$

A3 $\mathsf{sub}(\overline{a}, \overline{b}, \overline{c}) = \overline{d}$, where a and c are arbitrary strings of symbols, b is a symbol (or, equivalently, a length-1 string of symbols), and d is the string of symbols obtained from a by replacing all occurrences of the symbol b with c

A4 $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z} \big((\mathsf{x} ^\frown \mathsf{y}) ^\frown \mathsf{z} \big) = \big(\mathsf{x} ^\frown (\mathsf{y} ^\frown \mathsf{z}) \big)$

A5 $\forall \mathsf{x} \forall \mathsf{y} \big(\mathsf{x} ^\frown \mathsf{y} = \underline{0} \to \mathsf{x} = \underline{0} \land \mathsf{y} = \underline{0} \big)$

A6 $\forall \mathsf{x} \forall \mathsf{y} (\mathsf{x} ^\frown \mathsf{y} = \mathsf{x} \leftrightarrow \mathsf{y} = \underline{0}) \land \forall \mathsf{x} \forall \mathsf{y} (\mathsf{y} ^\frown \mathsf{x} = \mathsf{x} \leftrightarrow \mathsf{y} = \underline{0})$

A7 $\forall \mathsf{x} \forall \mathsf{y}\, \mathsf{sub}(\mathsf{x} ^\frown \overline{a}, \overline{a}, \mathsf{y}) = \mathsf{sub}(\mathsf{x}, \overline{a}, \mathsf{y}) ^\frown \mathsf{y}$, where a is a symbol

A8 $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z} \forall \mathsf{w}$
$\Big(\mathsf{x} ^\frown \mathsf{y} = \mathsf{z} ^\frown \mathsf{w} \leftrightarrow \exists \mathsf{v}_4 \big((\mathsf{x} = \mathsf{z} ^\frown \mathsf{v}_4 \land \mathsf{v}_4 ^\frown \mathsf{y} = \mathsf{w}) \lor (\mathsf{x} ^\frown \mathsf{v}_4 = \mathsf{z} \land \mathsf{y} = \mathsf{v}_4 ^\frown \mathsf{w}) \big) \Big)$

The problem with E is that, e.g., in A1

$$\overline{a} ^\frown \overline{b} = \overline{ab}, \textit{ where } a \textit{ and } b \textit{ are arbitrary strings of symbols}$$

we have an axiom for each pair of expressions a and b. From A1 we can prove

$$(\overline{a} ^\frown \overline{b}) ^\frown \overline{c} = \overline{a} ^\frown (\overline{b} ^\frown \overline{c});$$

but we cannot prove A4:

$$\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z} \big((\mathsf{x} ^\frown \mathsf{y}) ^\frown \mathsf{z}\big) = \big(\mathsf{x} ^\frown (\mathsf{y} ^\frown \mathsf{z})\big)$$

We cannot deal with all quotation constants in one proof.

The solution is to replace quotation constants with ‘more structural’ designators for expressions.

This is actually in keeping with our actual practice.

A first preliminary stab may look as follows:

DEFINITION

The symbols of $\mathcal{L}^-$ are:

(i) all variable symbols of $\mathcal{L}$,
(ii) all connectives, quantifiers, and auxiliary symbols of $\mathcal{L}$,
(iii) all function and predicate symbols of $\mathcal{L}$ and all constants that are not quotation constants,
(iv) the quotation constant $\overline{u}$ for each symbol u in clause (i), (ii), and (iii),
(v) the quotation constant $\underline{0}$.

The symbols in the first three clauses are called the basic symbols of $\mathcal{L}^-$. The remaining symbols, from clauses (iv) and (v), are called syntactic constants. An $\mathcal{L}^-$-expression is a finite string of $\mathcal{L}^-$-symbols.

This language has still infinitely many quotation constants because of the constants for the variables.

In real life the variables are generated from a finite alphabet using Arabic numerals.

We don't want to include Arabic numerals into our syntax theory. Instead we do the following:

Definition

A variable is an expression $(\mathsf{v}\cdots\mathsf{v})$ where $\mathsf{v}\cdots\mathsf{v}$ is a string consisting of the symbol v only. The variable containing exactly k-many occurrences of v is written as v_k:

$$\mathsf{v}_k := (\underbrace{\mathsf{v}\cdots\mathsf{v}}_{k}).$$

So we have only the single additional symbol v; we get rid of all the infinitely many variables.

Have we broken our promise not to employ any coding?

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Variables are now complex expressions.

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This causes additional problems and was the reason for the simplistic approach in E.

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Definition

The symbols of $\mathcal{L}^*$ are the following:

Basic symbols

(i) predicate symbols: sym (unary) and = (binary),
(ii) function symbols: q (unary), $\frown$ (binary), sub (ternary),
(iii) connectives $\neg$, $\rightarrow$, and the quantifier symbol $\forall$,
(iv) auxiliary symbols: parentheses (and) and symbols v and e,
(v) possibly finitely many other function, relation, and auxiliary symbols.

Syntactic symbols

(i) a constant $\underline{0}$
(ii) a constant $\underline{u}$ for each basic symbol u

There are only finitely many basic symbols and thus only finitely many symbols overall.

Generally, we like to keep the language open-ended. It's a little daft to have a predicate for truth or necessity, and then to have no vocabulary beyond the syntactic.

It would also be good to allow for objects beyond the syntactic. It's not a big problem, but requires some extra care: sorted or restricted quantifiers, and stipulation what happens when function symbols are applied to apples and chairs.

Definition

The $\mathcal{L}^*$-terms are defined as follows.

(i) All variables are terms.

(ii) If $t_1, \dots, t_n$ are terms and f is a function symbol of arity n, then $f t_1 \cdots t_n$ is a term.

A term is closed if and only if it contains no variables. A subterm of a term t is any term which is a subexpression of t.

Thus we have prefix notation: $\frown ab$ instead of $(a \frown b)$

Definition

The $\mathcal{L}^*$-formulæ are defined as follows.

(i) If $t_1, \ldots, t_n$ are terms and P is a predicate symbol of arity n, then $Pt_1 \cdots t_n$ is an atomic formula.

(ii) If φ is a formula and x is a variable, then $\forall x\, \varphi$ is a formula.

(iii) If φ and ψ are formulæ, so are $\neg\varphi$ and $(\varphi \to \psi)$.

Prefix notation also applies to predicate symbols: $= st$ instead of $s = t$.

We can prove various unique readability lemmata.

Definition

The quotation of a $*$-expression e is the $\mathcal{L}^*$-term $\overline{e}$ defined as follows.

(i) If e is the empty string, $\overline{e}$ is the expression $\underline{0}$.
(ii) If e is a basic symbol, $\overline{e}$ is the syntactic constant $\underline{e}$.
(iii) If e is a syntactic constant, $\overline{e}$ is the term $\mathsf{q}e$.
(iv) If $e \equiv fu$ is an expression of length at least 2 and u is a $*$-symbol, $\overline{e}$ is the term ${}^\frown\overline{f}\,\overline{u}$.

This is the same idea as for $\mathcal{L}^-$.

Definition

The pure terms are the terms of $\mathcal{L}^*$ generated by the following clauses.

(i) $\mathsf{q}\underline{0}$ is a pure term.

(ii) $\underline{u}$ and $\mathsf{q}\underline{u}$ are pure terms if u is a basic symbol.

(iii) if r is any pure term and s is a pure term of type (i) or (ii), then $\frown rs$ is a pure term.

Lemma

The pure terms are exactly the quotations of non-empty $*$-expressions.

Definition

$$\mathsf{Sing}(\mathsf{x}) \equiv \mathsf{x} \neq \underline{0} \wedge \forall \mathsf{w} \forall \mathsf{z}(\mathsf{x} = \mathsf{w}^\frown \mathsf{z} \rightarrow \mathsf{w} = \mathsf{x} \vee \mathsf{z} = \mathsf{x}).$$

An expression e for which $\mathsf{Sing}(\overline{e})$ holds is called a *singleton*. From the basic axioms of E^* we can deduce that every $*$-symbol is a singleton and that every singleton is either a basic symbol or a syntactic constant. The unary predicate symbol sym serves to demarcate the basic symbols from the syntactic symbols. As there are only finitely many $*$-symbols, the formula $\mathsf{sym}\,\mathsf{x}$ could be defined as a disjunction of equations $\mathsf{x} = \underline{u}$ where u ranges over the basic symbols of $\mathcal{L}^*$.

The following are all axioms of E^*. The minimal theory $\mathsf{E}^*_{\text{min}}$ of $\mathcal{L}^*$ comprises these axioms only.

Axioms for symbols

B1 $\mathsf{sym}\,\underline{u}$ for each basic $*$-symbol u

B2 $\forall \mathsf{x}\big(\mathsf{sym}\,\mathsf{x} \rightarrow \mathsf{Sing}(\mathsf{x})\big)$,

B3 $\forall \mathsf{x}\big(\mathsf{x} = \underline{0} \vee \mathsf{sym}\,\mathsf{x} \;\rightarrow\; \mathsf{Sing}(\mathsf{q}\mathsf{x})\big)$

B4 $\forall \mathsf{x}\big(\mathsf{Sing}(\mathsf{x}) \;\rightarrow\; \mathsf{sym}\,\mathsf{x} \vee \mathsf{x} = \mathsf{q}\underline{0} \vee \exists \mathsf{y}(\mathsf{sym}\,\mathsf{y} \wedge \mathsf{x} = \mathsf{q}\mathsf{y})\big)$

B5 $\forall \mathsf{x}\, \neg \mathsf{sym}(\mathsf{q}\mathsf{x})$,

B6 for each pair of distinct basic symbols u and v the sentence $\underline{u} \neq \underline{v}$

Axioms for concatenation

C1 $\forall \mathsf{x} \forall \mathsf{y}\big(\mathsf{x}^\frown \mathsf{y} = \underline{0} \;\rightarrow\; \mathsf{x} = \underline{0} \wedge \mathsf{y} = \underline{0}\big)$,

C2 $\forall \mathsf{x} \forall \mathsf{y}(\mathsf{x}^\frown \mathsf{y} = \mathsf{x} \leftrightarrow \mathsf{y} = \underline{0}) \wedge \forall \mathsf{x} \forall \mathsf{y}(\mathsf{x}^\frown \mathsf{y} = \mathsf{y} \leftrightarrow \mathsf{x} = \underline{0})$,

C3 $\forall \mathsf{x} \forall \mathsf{y} \forall \mathsf{z} \forall \mathsf{w}$

$$\Big(\mathsf{x}^\frown \mathsf{y} = \mathsf{z}^\frown \mathsf{w} \;\leftrightarrow\; \exists \mathsf{v}_4\big((\mathsf{x} = \mathsf{z}^\frown \mathsf{v}_4 \wedge \mathsf{v}_4{}^\frown \mathsf{y} = \mathsf{w}) \vee (\mathsf{x}^\frown \mathsf{v}_4 = \mathsf{z} \wedge \mathsf{y} = \mathsf{v}_4{}^\frown \mathsf{w})\big)\Big)$$

Axioms for quotation

D1 $\forall x \forall y \big(x \neq \underline{0} \wedge \mathrm{Sing}(y) \rightarrow q(x^\frown y) = \underline{\frown}^\frown qx^\frown qy \big)$,

D2 $\forall x \big(\mathrm{sym}\, x \rightarrow q(qx) = \underline{q}^\frown qx \big)$,

D3 $\forall x \forall y (qx = qy \rightarrow x = y)$

Axioms for substitution

E1 $\forall y \forall z \big(\mathrm{sub}(\underline{0}, y, z) = \underline{0} \wedge \mathrm{sub}(y, \underline{0}, z) = y \big)$

E2 $\forall x \forall y \forall z \big(y \neq \underline{0} \rightarrow \mathrm{sub}(x^\frown y, y, z) = \mathrm{sub}(x, y, z)^\frown z \big)$

E3 $\forall w \forall x \forall y \Big(\mathrm{Sing}(w) \wedge \forall z\, x^\frown w \neq z^\frown y \rightarrow \forall z\ \mathrm{sub}(x^\frown w, y, z) = \mathrm{sub}(x, y, z)^\frown w \Big)$

Axiom schema of induction

F1 the universal closure of $\forall x \big(\forall y \subset x\ \varphi(y) \rightarrow \varphi(x) \big) \rightarrow \forall x\, \varphi(x)$ for each formula $\varphi(x)$ of $\mathcal{L}^*$

I use the definition $x \subset y := \exists w \exists z (w^\frown x^\frown z = y) \wedge x \neq y$

Now we have to develop the syntax of our language $\mathcal{L}^*$ and define what a formula, proof etc is.

This proceeds via 'words'. We use the symbol e, which does not occur in words and can be used to separate words.

Using our strong syntax theory we can define various notion such as term, formula, etc of our language and prove expected observations.

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Coding

Now it’s time to look back at my promises ot provide a coding-free syntax theory.

Our feelings might tell us that Peano arithmetic is about numbers, while E^* is about expressions. That’s how I want you to see it.

One can think of Peano arithmetic as a syntax theory. It’s a syntax theory for expressions with one symbol, a stroke for instance. Thus the coding just tells us how to write down our expressions in a language with only one symbol.

E^* is closer to our normal notation, because we usually use more than one symbol. But it’s aslo not coding free. We don’t use e in our normal notation. Or do we? Is ‘space’ a symbol?

Also, in real life, the variables are not of the form $(v \cdots v)$. We write them with Arabic numerals as indices. Coding just means we abtract away from these differences. But such differences generate additional properties.

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Denotation

I will introduce theories of truth. For this I need denotion.

In general, the denotation of a term can depend on many things; but the denotions of terms of our syntax theory is ‘obvious’.

Let’s just introduce (define or axiomatize) a function symbol for ‘the denotation of’ or ‘the referent of’.

Theorem (denotation paradox)

Suppose $\mathsf{d}(x)$ *is a term of* $\mathcal{L}^*$ *and* E^* *derives the equation* $\mathsf{d}(\bar{s}) = s$ *for every closed term* s *of* $\mathcal{L}^*$*. Then* E^* *is inconsistent.*

Proof.

Apply the strong diagonal to the term $\underline{\forall} \frown \mathsf{d}(\mathsf{x})$:

$$\mathsf{E}^*_{\mathrm{min}} \vdash s = \underline{\forall} \frown \mathsf{d}(\bar{s}).$$

Assuming that $\mathsf{E}^* \vdash \mathsf{d}(\bar{s}) = s$, we deduce $\mathsf{E}^* \vdash s = \underline{\forall} \frown s$, from which it follows that $\mathsf{E}^* \vdash \underline{\forall} = \underline{0}$. $\square$

This looks like a accidental flaw of our syntax theory; it isn't.

There are recursion-theoretic results that prevent the evaluation function from being one of the functions being evaluated.

The Denotation paradox is very general. As long as we have strong diagonalization, there is no escape.

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In $\mathsf{E}^*_{\text{min}}$ we can define a denotation *formula* for terms in $\mathcal{L}^*_{\text{min}}$.

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Definition

The formula $\mathsf{Den}(x, y)$ is defined as

$$\exists z\Big(\mathsf{Seq}(z) \wedge \langle x, y\rangle \in_{\mathrm{seq}} z \wedge \forall w_0 \forall w_1\big(\langle w_0, w_1\rangle \in_{\mathrm{seq}} z \rightarrow \psi_{\mathsf{Den}}(w_0, w_1, z)\big)\Big),$$

where ψ_{Den} abbreviates

$$\psi_{\mathsf{Den}}(w_0, w_1, z) \;:=\; \big(\mathsf{Sing}(w_0) \wedge w_0 = \mathsf{q}w_1\big) \;\vee$$
$$\bigvee_{\mathsf{f} \in \mathcal{F}} \exists x_1 \cdots \exists x_{a_{\mathsf{f}}} \exists y_1 \cdots \exists y_{a_{\mathsf{f}}}\Big(w_0 = \underset{\cdot}{\mathsf{f}} x_1 \cdots x_k \wedge w_1 = \mathsf{f} y_1 \cdots y_{a_{\mathsf{f}}} \wedge \bigwedge_{i=1}^{a_{\mathsf{f}}} \langle x_i, y_i\rangle \in_{\mathrm{seq}} z\Big).$$

Lemma

The following are derivable in E^*.

(i) $\mathsf{Den}(\bar{s}, s)$ *for each closed term s,*

(ii) $\forall x\big(\mathsf{CTerm}(x) \to \exists y\, \mathsf{Den}(x, y)\big)$,

(iii) $\forall x\, \mathsf{Den}(\mathsf{q}x, x)$,

(iv) $\forall x \forall y\big(\mathsf{Den}(x, y) \to \mathsf{CTerm}(x)\big)$,

(v) $\forall x \forall y \forall z\big(\mathsf{Den}(x, y) \wedge \mathsf{Den}(x, z) \ \to\ y = z\big)$.

But we can pretend to have function symbol by using the following context definition:

$$\varphi(s^\circ) \;\equiv\; \mathsf{CTerm}(s) \land \exists y\big(\mathsf{Den}(s, y) \land \varphi(y)\big)$$

That is, there is no real function symbol $\cdots^\circ$, but we can express the denotation function using the formula $\mathsf{Den}(x, y)$. I find the (fake) functional notation easier to read.

Arithmetic

We can just concentrate on one symbol: Our theory $\mathsf{E}^*_{\text{min}}$ contains a theory of strings of vs.

$$\overline{n} := \underbrace{\overline{\mathsf{v}\cdots\mathsf{v}}}_{n}.$$

This induces a translation of arithmetic into our theory; and coding induces a translation into the opposite direction. They are intertranslatable.

Gödel

We can now prove both of Gödel's incompleteness theorems in E^*.

We can do so in a very pedestrian way of formalizing syntactic notions. This is the only way of getting the second theorem.

This now the place to pontificate about natural formalizations of provability.

Truth

We can define partial truth predicates. They do everything one would expect for formulæ with a fixed number of alternating quantifiers.

As soon as the want the unrestricted T-Schema, we run into the liar paradox, aka, Tarski's theorem on the undefinability of truth.

Long before primitivism became a big thing in philosophy, logicans understood that, if you cannot define something, just add it by brute force and axiomatize it.

I use the symbol $\square$ from before. This time we axiomatize it as truth.

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Long before primitivism became a big thing in philosophy, logicans understood that, if you cannot define something, just add it by brute force and axiomatize it.

I use the symbol $\Box$ from before. This time we axiomatize it as truth.

If E^* doesn't contain any axioms with $\Box$ beyond those in $\mathsf{E}^*_{\mathrm{min}}$, the following schema is consistent with E^*:

(1) $$\Box\overline{\varphi} \leftrightarrow \varphi \quad \text{for } \varphi \text{ a } \Box\text{-free sentence.}$$

The theory is called TB ('Tarski–biconditionals').

Some philosophers think that this is a good truth theory!

Note that we have induction with $\Box$.

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Theorem (conservativity of the typed T-schema)

Suppose E^* *is an extension of* $\mathsf{E}^*_{\min}$ *by* $\Box$*-free axioms. Then* TB *is a conservative extension of* E^* *for* $\Box$*-free formulas. That is, if* $\mathsf{TB} \vdash \varphi$ *and* φ *is* $\Box$*-free, then* $\mathsf{E}^* \vdash \varphi$.

$$(\mathrm{UTB}_\Gamma) \qquad \forall t_0 \cdots \forall t_{k-1}\big(\psi(t_0^\circ, \ldots, t_{k-1}^\circ) \leftrightarrow \Box\overline{\psi}[t_0, \ldots, t_{k-1}]\big)$$

Theorem

Suppose E^* *is an extension of* $\mathsf{E}^*_{\min}$ *by* $\Box$*-free axioms. If* $\mathsf{UTB} \vdash \varphi$ *and* φ *is* $\Box$*-free, then* $\mathsf{E}^* \vdash \varphi$*. In particular,* UTB *is consistent if* E^* *is.*

Definition

CT is the theory extending E^* by the following axioms:

CT1 $\forall s_1 \cdots \forall s_k \big(\Box(\underset{\cdot}{\mathsf{R}} s_1 \cdots s_k) \leftrightarrow \mathsf{R}(s_1^\circ, \ldots, s_k^\circ)\big)$ for each k and each predicate symbol $\mathsf{R} \in \mathcal{L}^*$ of arity k, excluding $\Box$,

CT2 $\forall \alpha \big(\mathsf{Sent}(\alpha) \to (\Box \underset{\cdot}{\neg} \alpha \leftrightarrow \neg \Box \alpha)\big)$

CT3 $\forall \alpha \forall \beta \Big(\mathsf{Sent}(\alpha \underset{\cdot}{\to} \beta) \to \big(\Box(\alpha \underset{\cdot}{\to} \beta) \leftrightarrow (\Box \alpha \to \Box \beta)\big)\Big)$

CT4 $\forall \alpha \forall v \Big(\mathsf{Sent}(\underset{\cdot}{\forall} v\, \alpha) \wedge \mathsf{Var}(v) \to \big(\Box \underset{\cdot}{\forall} v\, \alpha \leftrightarrow \forall s\, \Box \alpha[s/v]\big)\Big)$

Induction applies also to formulæ with $\Box$.

Lemma

UTB *is a subtheory of* CT.

Theorem

The global reflection principle for $\mathsf{E}^*_{\min}$ *is derivable in* CT. *That is,*

$$\mathsf{CT} \vdash \forall \alpha \big(\mathsf{Bew}_{\mathsf{E}^*_{\min}}(\alpha) \to \Box \alpha \big).$$

The provability predicate applies only to sentences.

We have now two primitive predicates $\Box$ and $\boxdot$.

Definition

KF is the extension of E^* by the following axioms:

KF1 $\forall s_1 \cdots \forall s_k \big(\Box \underset{\cdot}{\mathsf{R}} s_1 \cdots s_k \leftrightarrow \mathsf{R} s_1^\circ \cdots s_k^\circ\big)$ for each k and each relation symbol R of $\mathcal{L}_0^*$ of arity k,

KF2 $\forall s_1 \cdots \forall s_k \big(\boxdot \underset{\cdot}{\mathsf{R}} s_1 \cdots s_k \leftrightarrow \neg \mathsf{R} s_1^\circ \cdots s_k^\circ\big)$ for each k and each relation symbol R of $\mathcal{L}_0^*$ of arity k,

KF3 $\forall \alpha \big(\mathsf{Sent}(\alpha) \rightarrow (\Box \underset{\cdot}{\neg} \alpha \leftrightarrow \boxdot \alpha)\big)$,

KF4 $\forall \alpha \big(\mathsf{Sent}(\alpha) \rightarrow (\boxdot \underset{\cdot}{\neg} \alpha \leftrightarrow \Box \alpha)\big)$,

KF5 $\forall \alpha \forall \beta \Big(\mathsf{Sent}(\alpha \underset{\cdot}{\rightarrow} \beta) \rightarrow \big(\Box(\alpha \underset{\cdot}{\rightarrow} \beta) \leftrightarrow (\boxdot \alpha \vee \Box \beta)\big)\Big)$,

KF6 $\forall \alpha \forall \beta \Big(\mathsf{Sent}(\alpha \underset{\cdot}{\rightarrow} \beta) \rightarrow \big(\boxdot(\alpha \underset{\cdot}{\rightarrow} \beta) \leftrightarrow (\Box \alpha \wedge \boxdot \beta)\big)\Big)$,

KF7 $\forall \alpha \forall v \big(\mathsf{Sent}(\underset{\cdot}{\forall} v\, \alpha) \rightarrow (\Box \underset{\cdot}{\forall} v\, \alpha \leftrightarrow \forall s\, \Box \alpha[s/v])\big)$,

KF8 $\forall \alpha \forall v \big(\mathsf{Sent}(\underset{\cdot}{\forall} v\, \alpha) \rightarrow (\boxdot \underset{\cdot}{\forall} v\, \alpha \leftrightarrow \exists s\, \boxdot \alpha[s/v])\big)$,

KF9 $\forall s\, (\Box \underset{\cdot}{\Box} s \leftrightarrow \Box s^\circ) \wedge \forall s\, (\Box \underset{\cdot}{\boxdot} s \leftrightarrow \boxdot s^\circ)$,

KF10 $\forall s\, (\boxdot \underset{\cdot}{\Box} s \leftrightarrow \boxdot s^\circ) \wedge \forall s\, (\boxdot \underset{\cdot}{\boxdot} s \leftrightarrow \Box s^\circ)$.

Definition

FS is the theory which extends E^* by the following axioms and two rules of inference:

FS1 $\forall s_1 \cdots \forall s_k \big(\Box \underline{\mathsf{R}} s_1 \cdots s_k \leftrightarrow \mathsf{R}(s_1^\circ, \ldots, s_k^\circ)\big)$ for each k and each predicate symbol R of $\mathcal{L}^*_{\mathrm{syn}}$ of arity k;

FS2 $\forall \alpha \big(\mathsf{Sent}(\alpha) \to (\Box \underset{\cdot}{\neg} \alpha \leftrightarrow \neg \Box \alpha)\big)$;

FS3 $\forall \alpha \forall \beta \Big(\mathsf{Sent}(\alpha \underset{\cdot}{\to} \beta) \to \big(\Box(\alpha \underset{\cdot}{\to} \beta) \leftrightarrow (\Box \alpha \to \Box \beta)\big)\Big)$;

FS4 $\forall \alpha \forall v \big(\mathsf{Sent}(\underset{\cdot}{\forall} v\, \alpha) \to (\Box \underset{\cdot}{\forall} v\, \alpha \leftrightarrow \forall s\, \Box \alpha[s/v])\big)$;

FS5 NEC: if $\mathsf{FS} \vdash \varphi$ then $\mathsf{FS} \vdash \Box \overline{\varphi}$;

FS6 CoNEC: if $\mathsf{FS} \vdash \Box \overline{\varphi}$ then $\mathsf{FS} \vdash \varphi$.

De re modality