INVARIANCE CRITERIA FOR LOGICAL CONSTANTS

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Logic BPhil class

disclaimer: These are only some slides to aid my presentation. Negations may be added during my presentation. Please don't quote.

I aim at a definition of logical validity as a formula $x \vDash y$ with two variables where x is a set of formulæ and y a formula.

I am interested in formal languages.

For most of the time I confine myself to sentences to avoid talk of variable assignments.

A sentence φ is logically valid iff the argument with empty premiss set and φ as conclusion is valid.

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Semantic definition of validity

An argument is logically valid if, and only if (whenever all premisses are true under an interpretation of the non-logical vocabulary the conclusion is true under that interpretation).

Thus, the definition of logical validity (logical consequence, logical truth, etc) depends on the distinction between logical and non-logical vocabulary.

More and more philosophers of logic now doubt that there is a systematic way to make the distinction.

Criteria for logicality are applied 'hypothetically': If we had an expression with certain properties in our language would it be logical? My goal is a precise reconstruction of our mathematical, scientific, and philosophical reasoning in a formal language.

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I am mainly interested in a theory of logical consequence for my own language, not just toy languages. At the very least, the theory of logical consequence should apply to the language of first-order set theory with urelements.

The model-theoretic definition of consequence affords this: This definition applies to the language of set theory in which model-theoretic consequence is defined.

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Generality implies that logical constants behave on all objects in the same way (but there may be more to generality). Connectives and quantifiers should behave in the same way on all objects *and* formulæ. Invariance is used to make this precise.

Invariance under permutations harks back at least to (Mautner 1946, Tarski 1986) with roots in Kant & al.

There are numerous invariance criteria for logicality (Sher 1991, McGee 1996, Feferman 1999, 2010, Bonnay 2008, 2014, Casanovas 2007)…

Most (all?) are inspired by algebraic logic, and objects and operations are characterized as invariant, *not* expressions.

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In what follows D is always some non-empty set. D^ω is the set of all sequences of length ω of elements of D, i.e., variable assignments over D. For $a \in D^\omega$, a_n is the nth-member of the sequence (n-th projection).

Definition

A permutation π of D is a bijection of D onto itself. Thus a permutation is injective and surjective.

Abusing notation, for $a \in D^\omega$ I write $\pi(a)$ for the sequence $b \in D^\omega$ with elements $b_i = \pi(a_i)$ for all $i < \omega$.

Think of a predicate as the set of variable assignments that satisfy it.

Definition

A set $A \subseteq D^\omega$ is permutation-invariant iff
for all $a \in D^\omega$: $a \in A \Leftrightarrow \pi(a) \in A$.

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Assume $\mathcal{D}$ is a model with domain D.

Identity

The set $\{a \in D^{\omega} \colon \mathcal{D} \vDash x_1 = x_2\,[a]\}$ is permutation-invariant. 'Identity is logical.'

If $a_1 = a_2$, then $\pi(a_1) = \pi(a_2)$.

Connectives and quantifiers take formulæ and return new formulæ. Think of the extension of a formula as the set of variable assignments satisfying that formula in $\mathcal{D}$.

For any formula φ set $A_\varphi \coloneqq \{a \in D^\omega \colon \mathcal{D} \vDash \varphi[a]\}$.
Complication: There is no guarantee that for every set $A \subseteq D^\omega$ there is a φ such that $A = A_\varphi$. It depends on the language. People admit predicate symbols with infinitary arities, infinite conjunctions, and infinite quantifier blocks.

Let φ and ψ be formulæ. Then we have:

$$A_{\neg\varphi} = \{b \in D^\omega \colon \mathcal{D} \vDash \neg\varphi[b]\} = D \setminus A_\varphi$$

$$A_{\varphi\wedge\psi} = A_\varphi \cap A_\psi$$

$$A_{\exists v_n \varphi} = \{b \in D^\omega \colon \exists a \in A_\varphi \forall k\, (k \neq n \rightarrow a_k = b_k)\}$$

We consider unary operations from the power set of D^ω into itself $f\colon \mathcal{P}(D^\omega) \rightarrow \mathcal{P}(D^\omega)$ and binary operations $f\colon \mathcal{P}(D^\omega) \times \mathcal{P}(D^\omega) \rightarrow \mathcal{P}(D^\omega)$.

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Permutation invariance for unary operators

$f\colon \mathcal{P}(D^\omega) \to \mathcal{P}(D^\omega)$ is permutation-invariant iff
for all permutations π and $A \subseteq D^\omega$: $f(\pi(A)) = \pi(f(A))$.

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$f\colon \mathcal{P}(D^\omega) \times \mathcal{P}(D^\omega) \to \mathcal{P}(D^\omega)$ is permutation-invariant iff
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Example

$f_\neg\colon A \mapsto D \setminus A_\varphi$ is permutation-invariant.
$f_\wedge\colon (A, B) \mapsto A \cap B$ is permutation-invariant.
$f_{\exists v_n}\colon A \mapsto \{b \in D^\omega\colon \exists a \in A_\varphi \forall k\, (k \neq n \to a_k = b_k)\}$ is permutation-invariant.

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The operations corresponding to ∃, ∀, 'all *A*s are *B*s' are logical; 'Some tiger is *B*' isn't, if *D* contains tigers and non-tigers..

$\exists v_n\,(\text{Tiger}(v_n) \wedge \ldots)$ isn't logical because tigers can be mapped to non-tigers by π. We can make it and similar quantifiers logical by permitting only permutations that map tigers to tigers and not mapping non-tigers to tigers.

The same applies to second-order quantifiers.

Bold Thesis

The bold thesis is that permutation-invariance is logicality.

Problems:

(i) We have defined permutation invariance for sets of variable assignments and operations thereon, not for linguistic expressions. We need to explain what it means for an expression such as $\wedge$ to express $f_\wedge$.

(ii) So far we have defined permutation-invariance only relative to a non-empty set D. But logical constants behave in the same way on *all* objects not just those in D.

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'*A* and *B* and water=H_2O' (McGee 1996) necessarily has the same extension as '*A* and *B*'. Thus a permutation criterion cannot distinguish between them.

Cf. also the sentence letter *P* and $\perp$ (*falsum*). The latter should be a logical constant, the former shouldn't.

'The expression is only logical if there is not anything else to its meaning.'

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We need a definition of logicality that is absolute, not relative to a domain.

Solutions:

(i) Allow bijections between D_1 and D_2 in addition to permutations ('Sher–McGee'). This works if D_1 and D_2 have the same cardinality; but, if their cardinalities differ there are no bijections.

(ii) Consider surjective functions instead (giving up injectivity). See (Feferman 1999). Then non-identity, negation and conjunction are no longer logical. Feferman solved this by using functional type structures. See (Casanovas 2007).

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Solution: Reformulate the permutation-invariance criterion by considering permutations of all objects.

This is what Tarski (1986) may have had in mind, at least Williamson (1999) did. Cf. also (Friedman 1999).

Both use higher-order logic. I think this requires new (higher-order) objects (that cannot be in the domain of any permutation).

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