CLASSICAL DETERMINATE TRUTH

Volker Halbach

Firenze

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This is joint work with Kentaro Fujimoto.

I think we started in 2016.

The slides are on my web page.

We try to develop a theory of truth that slots into the place of the informal, but theoretical notion of truth that is used in philosophy and informal logical metatheory.

We do not claim that our theory is an analysis of the non-theoretical use of 'true'.

In mathematics the model-theoretic notion of truth is obviously what is needed.

If a belief is not true, it cannot be known. (‘Knowledge implies truth.’)

All true propositions (or sentences) can be verified.

There are sentences that a necessarily true, but cannot be known a priori.

There are true sentences that cannot be proved in Peano arithmetic.

If the premisses of a logically valid argument are true, its conclusion is also true.

Moral judgements are neither true nor false.

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We opt for classical logic.

We can do mathematics and science without a truth predicate, but without classical logic we would be in trouble.

The notion of truth in all these examples should be the same.

It is not sufficient to replace it with 'true in the standard model of arithmetic', 'truth restricted to Σ_{24}-sentences', or the like.

Philosophers have often focused on disquotation, i.e., the equivalence of $\mathsf{T}\ulcorner\varphi\urcorner$ and φ for sentences φ; but the compositional axioms are just as important – especially in philosophy.

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Fujimoto (2019, 2022) calls arguments like the following *blind deductions*:

> *Siegfried made a claim about the ring, which Brünhilde denied. Whatever Brünhilde said is true. Therefore Siegfried said something that is not true.*

$$\exists x\,(Sx \wedge B\underset{\cdot}{\neg}x)$$

$$\forall x\,(Bx \rightarrow Tx)$$

$$\therefore \exists x\,(Sx \wedge \neg Tx)$$

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$$\forall x\,(Bx \to Tx)$$

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\begin{aligned}
& \exists x\,(Sx \wedge B\underset{\cdot}{\neg}x) \\
& \forall x\,(Bx \to Tx) \\
& \forall x\,(T\underset{\cdot}{\neg}x \to \neg Tx) \\
\therefore\ & \exists x\,(Sx \wedge \neg Tx)
\end{aligned}
$$

The second premiss gives us $\forall x\,(B\underset{\cdot}{\neg}x \to T\underset{\cdot}{\neg}x)$.

The additional premiss, which is equivalent to $\neg\exists x\,(Tx \wedge T\underset{\cdot}{\neg}x)$, is the law of non-contradiction. T4 may be absent from other truth theories.

> *Smith believes a disjunction. He is justified in believing one disjunct, which happens to be false, while he does not believe the other disjunct, which happens to be true. Therefore Smith has a justified true belief.*

For this argument we need:

- Justification – just like provability – is closed under $\vee$-Intro.
- Truth is closed under $\vee$-Intro.

Fujimoto (2022):

> *Karl claimed that he saw someone near the scene of the murder and gave a description of the person (indicating that this person is the murderer). However, any person to whom Karl's description applies must have been heavier than 900 kg (according to, say, a credible report from an FBI forensic scientist). There is no one who weighs more than 900 kg. Karl made exactly one claim. Therefore, Karl's claim is not true.*

We need:

> *A universally quantified claim* $\forall x\, \varphi(x)$ *is true iff the formula* $\varphi(x)$ *without quantifier is satisfied by all objects.*

Closure of truth under logic is used all the time. Nobody would argue that knowledge isn't closed under logic, because truth isn't.

Moreover, truth is used in philosophy of logic: Proofs in calculus X preserve truth (without any typing).

Logical consequence preserves truth. See (Halbach 2020a).

We attribute truth to sentences rather than propositions.

The theory ought to be classical. That is, it ought to prove the compositional axioms.

These axioms in an unrestricted form are consistent with a theory of syntax – in contrast to disquotation.

We call a truth predicate (or its theory) *classical* if it satisfies the compositional axioms:

$$\text{T4} \quad \forall x \left(\mathrm{Sent}_T(x) \to (\mathsf{T}(\underset{\cdot}{\neg} x) \leftrightarrow \neg \mathsf{T}x)\right)$$
$$\text{T5} \quad \forall x \, \forall y \left(\mathrm{Sent}_T(x \underset{\cdot}{\wedge} y) \to (\mathsf{T}(x \underset{\cdot}{\wedge} y) \leftrightarrow \mathsf{T}x \wedge \mathsf{T}y)\right)$$
$$\text{T6} \quad \forall v \, \forall x \left(\mathrm{Sent}_T(\underset{\cdot}{\forall} v \, x) \to (\mathsf{T}(\underset{\cdot}{\forall} v \, x) \leftrightarrow \forall t \, \mathsf{T}x(t/v))\right)$$

… and so on for other connectives and quantifiers, if present.

I work in the language of arithmetic augmented with a predicate T (and soon also with D). PA with full induction in the augmented language is always assumed as base theory.

$\mathrm{Sent}_T(x)$ expresses that x is a sentence in the *full language*.

$\underset{\cdot}{\neg} x$ stands for 'the negation of x', similarly for $\underset{\cdot}{\wedge}$ and $\underset{\cdot}{\forall}$. $\forall v$ ranges over (codes of) variables, $\forall t$ over closed terms. $x(t/v)$ stands for 'the result of substituting t for v in x'.

Theories of classical and non-classical truth can both be formulated in classical logic. The Kripke–Feferman theory KF (Reinhardt 1986, Feferman 1991) is formulated in classical logic, but its truth predicate is not classical. Adding T4 to KF leads to inconsistency.

Also the supervaluational theories such as Cantini's (1990) VF are not classical; They may prove the truth of all classical tautologies, but the compositional axioms fail (and thus blind deductions).

We start with the consistent axioms T4 –T6 and then add disquotational axioms.

This is an inversion of the 'canonical approach' where we start with disquotation, 'transparency', or the like, and then, perhaps, add or derive compositional generalizations.

Often people try to maximize disquotation by endorsing principles such as $\mathsf{T}\ulcorner\mathsf{T}\ulcorner\varphi\urcorner\urcorner \leftrightarrow \mathsf{T}\ulcorner\varphi\urcorner$ for all φ and then try to see which compositional axioms are compatible with these principles.

What options concerning disquotation do we have if we stick to T4 –T6?

The following justifies the label ‘classical truth’:

Lemma

If Σ is classical (satisfies T4 –T6), we have Σ proves that all classical tautologies are true, that is, $\Sigma \vdash \forall x \left(\mathrm{Bew}_{\emptyset}(x) \wedge \mathrm{Sent}_{T}(x) \rightarrow \mathsf{T}x\right)$.

Call the sentence above *GRefl*(∅) (global reflection for logic).

The lemma imposes the following restriction on classical theories.

If Σ is classical, there is a finite subtheory of Σ that proves $GRefl(\varnothing)$. Let S be the conjunction of the axioms of this finite subtheory.

OBSERVATION

If $S \vdash \mathsf{T}\ulcorner S \urcorner$, Σ is classical and proves $\mathsf{T}\ulcorner\bot\urcorner \to \bot$, Σ is inconsistent.

PROOF

$$S \vdash \forall x \left(\mathrm{Bew}_{\varnothing}(x) \wedge \mathrm{Sent}_T(x) \to \mathsf{T}x\right)$$

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$$S \vdash \mathrm{Bew}_S(\ulcorner\bot\urcorner) \to \bot$$

$$S \vdash \neg\mathrm{Bew}_S(\ulcorner\bot\urcorner)$$

Thus S and therefore Σ are inconsistent by the second incompleteness theorem. □

Thus no finite subtheory of Σ can prove the truth of all axioms of Σ.

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Of course, we will want $\mathsf{T}\ulcorner\varphi\urcorner \leftrightarrow \varphi$ for sentences φ without T.

The only better known classical theory is FS (Friedman and Sheard 1987, Halbach 1994). In FS disquotation is captured by rules:

From φ conclude $\mathsf{T}\ulcorner\varphi\urcorner$ and *vice versa*.

The above problem is avoided, because there is no finite subtheory $S \subseteq \mathsf{FS}$ with $S \vdash \mathsf{T}\ulcorner S\urcorner$. FS proves the truth of every finite subtheory, but no finite subtheory of FS does.

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We need semantic ascent and descent (Quine 1970).

For every suitable sentence φ we create a copy $\mathsf{T}\ulcorner\varphi\urcorner$. The copies now allow us to generalize:

All sentences of the form 'n=n' are true.

That is, '$0=0$' *is true,* '$1=1$' *is true,* '$2=2$' *is true,*…

That is, $0=0$, $1=1$, $2=2$,…

Iterating this, permits more and more generalizations.

We cannot form copies of all sentences, that is, we cannot semantically ascend from them. Some are already at the highest level of generality – such as T4 –T6.

We add a predicate D that applies to sentences that are open to semantic ascent.

The sentences of the base language together with those needed for semantic ascent and generalization form the set $\mathfrak{D}$.

For each sentence φ in $\mathfrak{D}$, we also add an equivalent new sentence $\mathsf{T}\ulcorner\varphi\urcorner$ as a 'copy' of φ. These copies are required for semantic ascent.

We need to close under connectives and quantifiers.

For instance, $\forall x\,(x = \ulcorner 0 = 0\urcorner \rightarrow \mathsf{T}x)$ should be in $\mathfrak{D}$, because all instances $\ulcorner\varphi\urcorner = \ulcorner 0 = 0\urcorner \rightarrow \mathsf{T}\ulcorner\varphi\urcorner$ are in $\mathfrak{D}$, although some instances of $\mathsf{T}\ulcorner\varphi\urcorner$ are not.

When $\ulcorner\varphi\urcorner = \ulcorner 0 = 0\urcorner$ is false, it's sufficient that $\ulcorner\varphi\urcorner = \ulcorner 0 = 0\urcorner$ is in $\mathfrak{D}$. When $\ulcorner\varphi\urcorner = \ulcorner 0 = 0\urcorner$ is true, it's sufficient that $\mathsf{T}\ulcorner 0 = 0\urcorner$ is in $\mathfrak{D}$.

DETERMINATENESS OF CONDITIONALS

$\varphi \rightarrow \psi$ is in $\mathfrak{D}$ iff either (both φ and ψ are in $\mathfrak{D}$) or (φ is false and in $\mathfrak{D}$) or (ψ is true and in $\mathfrak{D}$).

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D is to be read as 'open to semantic ascent'.

D1 $\forall s\,\forall t\,\mathsf{D}s\dot{=}t$

D2 $\forall t\,\big(\mathsf{D}\dot{\mathsf{T}}t \leftrightarrow \mathsf{D}t^{\circ}\big)$

D3 $\forall t\,(\mathsf{D}\dot{\mathsf{D}}t \leftrightarrow \mathsf{D}t^{\circ})$

D4 $\forall x\,\big(\mathrm{Sent}_T(x) \rightarrow (\mathsf{D}(\dot{\neg}x) \leftrightarrow \mathsf{D}x)\big)$

D5 $\forall x\,\forall y\Big(\mathrm{Sent}_T(x\dot{\wedge}y) \rightarrow \big(\mathsf{D}(x\dot{\wedge}y) \leftrightarrow \big((\mathsf{D}x \wedge \mathsf{D}y) \vee (\mathsf{D}x \wedge \mathsf{F}x) \vee (\mathsf{D}y \wedge \mathsf{F}y)\big)\big)\Big)$

D6 $\forall v\,\forall x\Big(\mathrm{Sent}_T(\dot{\forall}v\,x) \rightarrow \big((\mathsf{D}(\dot{\forall}v\,x) \leftrightarrow \big(\forall t\,\mathsf{D}x(t/v) \vee \exists t\big(\mathsf{D}x(t/v)\wedge\mathsf{F}x(t/v)\big)\big)\big)\Big)$

$\mathsf{F}x$ abbreviates $\mathsf{T}\dot{\neg}x$, which expresses the falsity of x.

For truth we need only one additional axiom beyond T4 –T6:

DDS $\forall t_1 \ldots \forall t_n \left(\mathsf{D}\ulcorner\varphi(\dot{t}_1, \ldots, \dot{t}_n)\urcorner \rightarrow \left(\mathsf{T}\ulcorner\varphi(\dot{t}_1, \ldots, \dot{t}_n)\urcorner \leftrightarrow \varphi({t_1}^\circ, \ldots, {t_n}^\circ)\right)\right)$

This is disquotation (or semantic ascent and descent) for determinate (=generalizable) sentences.

DDS is equivalent to the following three axioms:

T1 $\forall s\, \forall t \left(\mathsf{T} s \dot{=} t \leftrightarrow s^\circ = t^\circ\right)$

T2 $\forall t (\mathsf{D} t^\circ \rightarrow \mathsf{T}\dot{\mathsf{D}} t)$

T3 $\forall t \left(\mathsf{D} t^\circ \rightarrow (\mathsf{T}\dot{\mathsf{T}} t \leftrightarrow \mathsf{T} t^\circ)\right)$

THE AXIOMS OF CD

The theory CD is given by the axioms of PA and the following axioms:

T1 $\forall s\, \forall t\, \big(\mathsf{T} s \underset{\cdot}{=} t \leftrightarrow s^\circ = t^\circ\big)$

T2 $\forall t (\mathsf{D} t^\circ \rightarrow \mathsf{T} \underset{\cdot}{\mathsf{D}} t)$

T3 $\forall t\, \big(\mathsf{D} t^\circ \rightarrow (\mathsf{T} \underset{\cdot}{\mathsf{T}} t \leftrightarrow \mathsf{T} t^\circ)\big)$

T4 $\forall x\, \big(\mathrm{Sent}_T(x) \rightarrow (\mathsf{T}(\underset{\cdot}{\neg} x) \leftrightarrow \neg \mathsf{T} x)\big)$

T5 $\forall x\, \forall y\, \big(\mathrm{Sent}_T(x \underset{\cdot}{\wedge} y) \rightarrow (\mathsf{T}(x \underset{\cdot}{\wedge} y) \leftrightarrow \mathsf{T} x \wedge \mathsf{T} y)\big)$

T6 $\forall v\, \forall x\, \big(\mathrm{Sent}_T(\underset{\cdot}{\forall} v\, x) \rightarrow (\mathsf{T}(\underset{\cdot}{\forall} v\, x) \leftrightarrow \forall t\, \mathsf{T} x(t/v))\big)$

D1 $\forall s\, \forall t\, \mathsf{D} s \underset{\cdot}{=} t$

D2 $\forall t\, \big(\mathsf{D} \underset{\cdot}{\mathsf{T}} t \leftrightarrow \mathsf{D} t^\circ\big)$

D3 $\forall t\, (\mathsf{D} \underset{\cdot}{\mathsf{D}} t \leftrightarrow \mathsf{D} t^\circ)$

D4 $\forall x\, \big(\mathrm{Sent}_T(x) \rightarrow (\mathsf{D}(\underset{\cdot}{\neg} x) \leftrightarrow \mathsf{D} x)\big)$

D5 $\forall x\, \forall y\, \Big(\mathrm{Sent}_T(x \underset{\cdot}{\wedge} y) \rightarrow \big(\mathsf{D}(x \underset{\cdot}{\wedge} y) \leftrightarrow \big((\mathsf{D} x \wedge \mathsf{D} y) \vee (\mathsf{D} x \wedge \mathsf{F} x) \vee (\mathsf{D} y \wedge \mathsf{F} y)\big)\big)\Big)$

D6 $\forall v\, \forall x\, \Big(\mathrm{Sent}_T(\underset{\cdot}{\forall} v\, x) \rightarrow \big((\mathsf{D}(\underset{\cdot}{\forall} v\, x) \leftrightarrow \big(\forall t\, \mathsf{D} x(t/v) \vee \exists t\big(\mathsf{D} x(t/v) \wedge \mathsf{F} x(t/v)\big)\big)\big)\Big)$

Axiom T2 can be strengthened to a biconditional:

T2$^+$ $\forall t(\mathsf{D}t^\circ \leftrightarrow \mathsf{T}\underset{\cdot}{\mathsf{D}}t)$

We endorse the following two axioms, but they are irrelevant for all later claims.

R1 $\forall x\, \forall v\, \forall s\, \forall t \left(\left(\mathrm{Sent}_T(\underset{\cdot}{\forall} v\, x) \wedge s^\circ = t^\circ \right) \rightarrow \left(\mathsf{T}x(s/v) \leftrightarrow \mathsf{T}x(t/v) \right) \right)$

R2 $\forall x\, \forall v\, \forall s\, \forall t \left(\left(\mathrm{Sent}_T(\underset{\cdot}{\forall} v\, x) \wedge s^\circ = t^\circ \right) \rightarrow \left(\mathsf{D}x(s/v) \leftrightarrow \mathsf{D}x(t/v) \right) \right)$

CD is ω-consistent.

Let λ be a liar sentence.

$\mathsf{CD} \nvdash \lambda$ and $\mathsf{CD} \nvdash \neg\lambda$

$\mathsf{CD} \vdash \mathsf{T}\ulcorner\lambda \vee \neg\lambda\urcorner$, but $\mathsf{CD} \nvdash \mathsf{T}\ulcorner\mathsf{T}\ulcorner\lambda \vee \neg\lambda\urcorner\urcorner$

$\mathsf{CD} \nvdash \mathsf{T}\ulcorner(\mathsf{T}\ulcorner\neg\lambda\urcorner \leftrightarrow \neg\mathsf{T}\ulcorner\lambda\urcorner)\urcorner$ Not all axioms are provably true in CD.

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CD is ω-consistent.

Let λ be a liar sentence.

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Liar sentences are not determinate.

MAXIMAL GENERALITY

The compositional axioms such as $\forall x \left(\mathrm{Sent}_T(x) \to (\mathsf{T}(\underset{\cdot}{\neg} x) \leftrightarrow \neg \mathsf{T} x)\right)$ are not determinate. We do not have semantic ascent for T4 – T6. The compositional axiom are already 'maximally general'; they cannot be further generalized. There is no semantic ascent for such sentences. They also cannot be reached by semantic ascent, because they generalize over *all* sentences, including themselves and liar sentences.

This way we defuse the observation above for (type-free) classical truth predicates. The truth of T4 is not provable in CD.

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This way we defuse the observation above for (type-free) classical truth predicates. The truth of T4 is not provable in CD.

Reinhardt (1986) and Bacon (2015) have posed a very general problem for truth theories in the style of CD (here slightly simplified):

Reinhardt's problem

Assume $S \vdash \mathsf{D}\ulcorner\varphi\urcorner \to (\mathsf{T}\ulcorner\varphi\urcorner \leftrightarrow \varphi)$ for all φ, then there is a sentence γ such that $S \vdash \gamma$ and $S \vdash \neg\mathsf{D}\ulcorner\gamma\urcorner$

So S proves that some of its theorems are not determinate!

Indeterminate sentences can be meaningful and provable. D should be read as 'open to semantic ascent'. One should not think of D as 'unproblematic' or 'meaningful' or the like.

PROOF-THEORETIC STRENGTH

CD is as strong as $\mathrm{RT}_{<\varepsilon_0}$ or KF.

CD^+ is CD with $\mathrm{T2}^+$, that is, $\forall t(\mathsf{D}t^\circ \leftrightarrow \mathsf{T}\dot{\mathsf{D}}t)$ instead of only $\forall t(\mathsf{D}t^\circ \rightarrow \mathsf{T}\dot{\mathsf{D}}t)$.

PROOF-THEORETIC STRENGTH

CD^+ is as strong as $\mathrm{RT}_{<\varepsilon_{\varepsilon_0}}$ or CT(KF).

Adding the minimality schemata to CD for determinateness gives ID_1. It can also be added to CD^+ and a still stronger system is obtained. Minimality permits us, e.g., to prove that truth teller sentences are indeterminate.

T and D

T cannot be defined from D in an obvious way; and there is no straightforward definition of D in terms of T (unlike, say, in KF); but Nicolai suggested to define Dx as $(\mathsf{T}\underset{\cdot}{\mathsf{T}}x \lor \mathsf{T}\underset{\cdot}{\mathsf{F}}x) \land \neg(\mathsf{T}\underset{\cdot}{\mathsf{T}}x \lor \mathsf{T}\underset{\cdot}{\mathsf{F}}x)$.

CD provides an example of a theory whose strength depends on the interaction of two predicates. Removing the axioms specific to D (whatever that means) makes the theory very weak.

The predicates have no ‘relative typing’. T is applied to sentences with D in a non-trivial way, and vice versa to obtain the full proof-theoretic strength. Omitting interaction axioms weakens the system.

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The determinateness axioms capture an inductive definition. Nothing in the determinateness axioms forces the minimal fixed point. So CD cannot rule out that e.g. the truth teller is determinate.

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In (2020a, 2020b) I define logical consequence as truth preservation under all interpretations. Truth is understood in the sense of CD over set theory instead of arithmetic.

I obtain (more or less) standard classical consequence.

With KF or the like instead of CD I would obtain non-classical consequence.

With model-theoretic consequence I lack the intended interpretation, that is, I cannot interpret sentences by themselves. The homophonic interpretation is admitted on my account.

The notion of consequence is also universal. It applies to all sentences, including those with D and T. This requires a type-free truth predicate.

We can prove soundness for logic in this system.

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We can prove soundness for logic in this system.

We could try to add $\forall t\, \mathsf{D}\dot{\mathsf{D}}t$, so all T-free sentences become determinate. Is the result consistent? If so, how strong is it?

SUMMARY

CD is a theory of classical truth. The compositional axioms T4 –T6 hold in an unrestricted form and, therefore, blind deductions can be carried out in CD without any further limitations.

Thus the internal logic of CD is fully classical. It permits applications to the theory of logical consequence etc.

Disquotation is restricted to sentences that are open to semantic ascent. The axioms like T4 –T6 themselves and the liar sentence are already fully general and not open to semantic ascent.

The resulting theory is ω-consistent unlike FS.

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