Logical Constants

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Firenze
31st October 2023

The plan

The Full Project

Logic

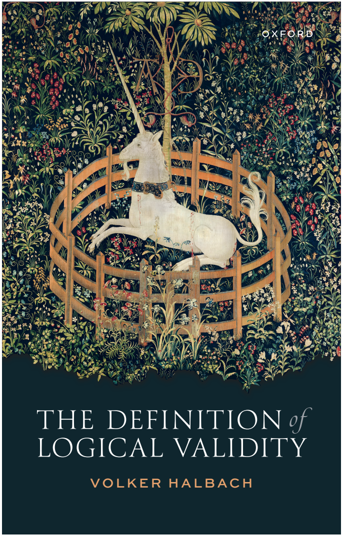
Subject-Specificity

Permutation Invariance and Models

the classification

Unicorns

The Full Project



The Definition of Logical Validity, Oxford University Press, 2025

Unrestricted Quantification and Logical Constants, *Journal for the Philosophy of Mathematics* 1 (2024), 123–140

I am interested mainly in formal languages, more specifically, finite expansions of the language of set theory.

Logic

Semantic characterization of validity

An argument is logically valid if, and only if whenever all premisses are true under an interpretation of the non-logical vocabulary the conclusion is true under that interpretation.
A sentence is logically valid iff it is true under all interpretations of the non-logical vocabulary.

There is the logical vocabulary (aka logical constants) and the non-logical vocabulary (and perhaps auxiliary symbols such as the brackets).

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The most popular way to turn the characterization into a definition is the model-theoretic account of logical consequence.

The distinction between logical and non-logical vocabulary is baked in the model theory. A model $\mathcal{M}$ provides interpretations of the predicate symbols as well as function symbols (including individual constants), but not of $\wedge$ or $\neg$.

The treatment of the identity symbol can be varied.

In a way, the standard first-order quantifiers $\forall$ and $\exists$ do receive an interpretation by $\mathcal{M}$ (namely the *domain*). However, their re-interpretation is limited: $\exists x$ cannot be interpreted as 'at least five x'.

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What about generalized quantifiers such as 'There are more *A*s than *B*s', 'There are infinitely many', 'There are $\aleph_7$ many'?

What about second-order quantifiers?

What about modal operators?

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consequence of the semantic characterization
If a sentence φ contains only logical constants, then φ is logically valid if it is true (and it's a logical contradiction if false).

Potential examples: $\neg\bot$, $\forall x\, x = x$, $\exists x \exists y\, \neg x = y$, $\forall X\, \forall y\, (Xy \vee \neg Xy)$

There is a wide-spread view that the choice of logical constants is purely pragmatic. I have concerns.

(i) Whether logicism can be successful becomes a matter of choice.

(ii) At least some indeterminacy problems (Skolem's paradox, multiversism about arithmetic) arise from model theory (and therefore from the choice of logical constants).

(iii) Debates about 'the correct logic' depend on the logical constants. (We could have intuitionistic or strong Kleene connectives as logical constants, but LEM (without exception) as another kind of truth.)

(iv) The metatheoretic properties of logical consequence (compactness, decidability, enumerability, arithmetical definability, axiomatizability) depend on the choice of logical constants.

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Of course, we can permit variations of interpretations in arbitrary ways to obtain other notions of consequence.

We can re-interpret even the connectives. Cf. Carnap categoricity.

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Subject-Specificity

A logical term should not be subject-specific; *it should behave on all objects in the same way.*

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History

Usually, (Tarski 1986a, Mautner 1946, Mostowski 1957) are cited and then there was another surge from around 1990: (Sher 1991, Feferman 1999, McGee 1996, Bonnay 2008).

Some aspects are foreshadowed in Tarski (1935). I am sure there are earlier philosophical and mathematical accounts (long before algebraic logic). E.g., the observation that intersection is invariant under permutations must be old.

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A *permutation* of a set S is an injective mapping from S onto itself. I.e. permutation replaces each object in S with a (not necessarily different) object. That is, every element of D is assigned exactly one proxy from D; different objects have different proxies; and everything in D is a proxy of something.

Satisfaction conditions change for non-logical expressions: 'is white' is satisfied by snow; but if snow is replaced with coal as proxy, it's no longer satisfied.

The predicate that applies to everything is logical: It applies to me as well as to all objects (proxies) I can be replaced with.

The formula $x = y$ is satisfied by VH and VH. If I am replaced with something else (a proxy), it's still satisfied. $x = y$ is not satisfied by VH and Trump. If they are replaced with proxies, it's still not satisfied (here we need injectivity).

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Permutation Invariance and Models

Think of a first-order model $\mathcal{M}$ as an ordered pair $\langle D, I\rangle$ consisting in a domain D and an interpretation of the predicate symbols (forget about function symbols for the moment being).

The interpretation of some symbols is kept fixed, e.g., the interpretation of $=$.

As above, a permutation π of D is an injective mapping of D onto D.

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Given a variable assignment a over D and a function π on D, I write $\pi'(a)$ for the variable assignment b with $b(i) = \pi(a(i))$ for $i \in \omega$.

Definition

A formula Rv_0v_1 is *permutation invariant over* $\mathcal{M}$ iff for all permutations π of D and variable assignments a over D: $(\mathcal{M} \vDash Rv_0v_1[a]$ iff $\mathcal{M} \vDash Rv_0v_1[\pi'(a)])$.

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Also, which models should be considered?

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A term can behave in each model logically, but very differently in different models.

Imagine a predicate W that applies to all objects in models with a domain containing at least one wombat and to nothing in all other domains. W is permutation invariant in the model-theoretic sense!

Similarly, 'wombat conjunction' functions like $\wedge$ on domains containing wombats, but like $\vee$ on domains without wombats. (Cf. McGee 1996.)

A binary predicate permutation invariant in the model-theoretic sense can be have like identity in some models and like distinctness in others and apply to everything in still others.

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Mostowski (1957) was aware of the problem and considered injective and surjective mappings between domains. See also (Sher 1996).

The problem is that one can force R to behave in the same way on all domains *of the same cardinality*. Between models of different cardinalities R can behave in wildly different ways.

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Return to the original idea and ditch domains!

This will solve the wombat problem. We consider only what is the case and not different interpretations (models).

I suspect the reason for using models in the criterion for logicality is some sort of reductionism.

This is by no means a new idea. Already Tarski (1986b, p. 149) may have considered both options, i.e., defining invariance over elements of a specific domain as well as over all objects:

> *we would consider the class of all one-one transformations of the space, or universe of discourse, or 'world', onto itself.*

Also Williamson (1999) seems to make use of a formulation of the permutation-invariance criterion without domains.

DOMAIN-BASED INVARIANCE

A term or operation is invariant (and thus supposed to be logical) iff the term is invariant for all domains *D* under all permutations of *D*.

is replaced with

DOMAIN-FREE INVARIANCE

A term or operation is invariant (and thus supposed to be logical) iff the term is invariant under arbitrary permutations of the entire universe.

For the latter we need *satisfaction over the entire universe* and *permutations of the universe*. This is why I need to move to higher-order logic or use satisfaction predicate.

We deal with variable assigments as our extensions of formulæ. Because all our formulæ are finite, their extensions are finitary in the following sense:

Definition

A class $A \subseteq V^{\omega}$ of variable assignments is finitary iff there is a finite set $I \subset \omega$ such that $\forall b \left(\exists a \in A \, \forall i \in I \; b(i) = a(i) \;\rightarrow\; b \in A \right)$.

V^{ω} is the class of all functions from ω into the universe V, i.e., the class of all variable assignments.

$\mathcal{F}$ is the class (3rd order) of all finitary classes of variable assignments.

Under reasonable assumptions, the extension $|\varphi|$ of any formula will thus be finitary (in contrast to McGee 1996).

Lemma

For any formula with finitely many variables,
$|\varphi| := \{a \in V^{\omega} \colon \mathsf{Sat}(\ulcorner\varphi\urcorner, a)\}$ is a finitary variable assignment.

Sat is the definition of satisfaction for the first-order language in the second-order language in the style of Tarski.

If φ is satisfied by at least one variable assignment, $|\varphi|$ is a proper class.

We get the picture similar to that obtained in algebraic logic:

If φ is a true sentence (without free variables), we have $|\varphi| = V^{\omega}$; if φ is false, we have $|\varphi| = \emptyset$.

$|\neg\varphi| = V^{\omega} \setminus |\varphi|$

$|\varphi \wedge \psi| = |\varphi| \cap |\psi|$

If connectives are treated as truth functions, we don't do justice to the fact that connectives can be applied to open formulæ.

The operation expressed by existential quantification with $\exists v_k$ is called cylindrification:
$|\exists v_k \varphi| = \{b \in V^{\omega}\colon \exists a \in |\varphi|\ \forall i \neq k\ a(i) = b(i)\}$.

We can consider generalized quantifiers etc in the same way.

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Definition

For $n \geq 0$ an n-ary operation O is a function that maps every n-tuple $\langle A_0, \ldots, A_{n\text{-}1} \rangle$ of finitary classes of variable assignments to a class of finitary variable assignments A.

Formulæ and thus also atomic formulæ correspond to 0-ary operations, i.e., O has an output (the extension), but no input.

The operation of negation is a unary operation. It maps a finitary class A of variable assignments to the complement $V^\omega \setminus A$.

The operation of conjunction is binary and maps $\langle A_1, A_2 \rangle$ to $A_1 \cap A_2$.

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I work in some class theory.

Definition

A permutation of V is an injective mapping of V onto V. The permutation Π' of variable assignments induced by a permutation Π of V is the class-sized function mapping every $a \in V^\omega$ to the variable assignment $b \in V^\omega$ such that $b(i) = \Pi(a(i))$ for all $i \in \omega$.

As usual, I conflate permutations of V and the permutations induced by it and write Π where I should write Π'.

Definition

An n-ary operation O is permutation-invariant iff for all permutations Π and all $A_i \in \mathcal{F}$ with $i < n$:

$$O(\Pi(A_1), \ldots, \Pi(A_{n\text{-}1})) = \Pi(O\langle A_0, \ldots, A_{n\text{-}1}\rangle)$$

Operations in this sense are operations on proper classes.

Consider the 0-place operation $O_= := |v_0 = v_1|$. We have:

$$O_= = \Pi(O_=)$$

Identity, distinctness, V^ω are all permutation-invariant, as are negation (complementation), conjunction (intersection), existential (cylindrification) and universal quantification.

For conjunction we have for all (finitary) classes A and B of variable assignments:

$$\begin{aligned} O_\wedge(\Pi(A), \Pi(B)) &= (\Pi(A) \cap \Pi(B)) \\ &= \Pi(A \cap B) \\ &= \Pi(O_\wedge(A, B)) \end{aligned}$$

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Definition (expressing an operation)

Assume that $\circ$ is a predicate symbol or an n-ary connective or quantifier and define $|\varphi| := \{a \in V^\omega \colon \mathsf{Sat}(\ulcorner\varphi\urcorner, a)\}$ as above. Then $\circ$ expresses an operation O that maps $\langle|\varphi_1|, \ldots, |\varphi_n|\rangle$ to $|\circ(\varphi_1, \ldots, \varphi_n)|$ for all first-order formulæ of the chosen language.

Definition (logical term)

A term (predicate symbol, connective, quantifier) is logical iff is expresses a permutation-invariant operation.

Note that $\exists v_7$ is a term that is different from $\exists v_8$.

Thus $=$, $\neg$, $\wedge$, $\rightarrow$, $\exists v_k$, and $\forall v_k$ are all logical terms, while $\in$, 'there is at least one ordinal' (if present as prim. quantifier) etc. are not.

Our classification of logical constants (as well as our theory of logical consequence) should be stable in the sense that it does not produce strange results when the language is expanded.

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McGee (1992) considered the quantifier $\exists^{AI} v_k$ expressing that there are absolutely infinitely many. This quantifier corresponds to the operation that maps $A \in \mathcal{F}$ to the class B of all variable assignment such that

$$b \in B \text{ iff } (\{a \in A{:}\forall i \neq k\ a(i) = b(i)\} \text{ is a proper class.})$$

On a straightforward domain-relative formulation of the permutation-invariance criterion, the operation will also qualify as permutation-invariant, but in a trivial way because there is no proper class of variable assignments over a set-sized domain.

On a domain-relative formulation, the quantifier ‘there are unboundedly many ordinals’ comes out as a logical term.

At the beginning I mentioned the following principle:

consequence of the semantic characterization
If a sentence φ contains only logical constants, then φ is logically valid if it is true (and it's a logical contradiction if false).

Thus, $\exists^{AI}v_k\ v_k = v_k$ will then qualify as a logical contradiction on a domain-relative account. It is logically true on my account.

the classification

On my domain-free account of permutation invariance as criterion for logical constants, the following problems are solved:

(i) I obtain principled reasons to classify $=$, $\wedge$, $\neg$, $\exists v_k$ as logical terms.

(ii) The logicality of the connectives isn't trivial in the same way as for functional type structures.

(iii) Under certain assumptions, $\Box$ will not be a logical term.

(iv) The problem of wombat conjunction is solved.

(v) The McGee quantifier $\exists^{AI} v_k$ is treated adequately.

(vi) Cardinality quantifiers, the Härtig quantifier are classified as logical constants.

(vii) The higher-order quantifiers can be eliminated using a satisfaction predicate (as in the book).

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Unicorns

My entire story is extensional. $v_0 \neq v_0$ and 'v_0 is a unicorn' are extensionally equivalent. Both come out as logical constants.

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Permutation Invariance and Higher-Order Logic

Usually, permutation invariance is defined for objects in and operations between type structures. See (Gómez-Torrente 2002, Bonnay 2008).

One starts with a permutation of the domain, which is then lifted to higher-order objects. On this approach higher-order existential and universal quantifiers are logical constants.
Griffiths and Paseau (2016, p. 492) write:

> *The isomorphism invariance of second-order quantifiers shows that they are topic-neutral and general in the same way as first-order quantifiers. In this way, isomorphism invariance can be used to defend second-order logic as logic (the first of our paper?s two subsidiary aims), in a way that dovetails with pluralist justifications of second-order quantification as logical but does not presuppose them.*

But the permutation is not permitted to replace a first-order object with a higher-order object etc. Permutation invariance doesn't show anything about higher-order quantification.

Feferman's Stricter Standards

BTW in functional type structures connectives are treated as acting on sentences only. This leads to odd results.

Here we can deal with arbitrary formulæ. Feferman (1999) writes:

> *Finally, as pointed out to me by Bonnay, it is hard to see how identity could be determined to be logical or not by a set-theoretical invariance criterion of the sort considered here, since either it is presumed in the very notion of invariance itself that is employed – as is the case with invariance under isomorphism or one of the partial isomorphism relations considered in the next section – or it is eliminated from consideration as is the case with invariance under homomorphism.*

Distinctness is not invariant if injectivity is dropped, as Feferman (1999) suggested. In the setting here then, however, negation is no longer invariant (cf. Casanovas 2007). That's why we need to treat connectives as acting on variable assignments.

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References

Denis Bonnay. Logicality and invariance. *Bulletin of Symbolic Logic*, 14:29–68, 2008.

Enrique Casanovas. Logical operations and invariance. *Journal of Philosophical Logic*, 36:33–60, 2007.

Solomon Feferman. Logic, logics, and logicism. *Notre Dame Journal of Formal Logic*, 40:31–54, 1999.

Mario Gómez-Torrente. The problem of logical constants. *Bulletin of Symbolic Logic*, 8:1–37, 2002.

Owen Griffiths and Alexander C Paseau. Isomorphism invariance and overgeneration. *Bulletin of Symbolic Logic*, 22:482–503, 2016.

Friedrich I. Mautner. An extension of Klein's Erlanger program: Logic as invariant-theory. *American Journal of Mathematics*, 68:345–384, 1946.

Vann McGee. Two problems with Tarski's theory of consequence. *Proceedings of the Aristotelian Society*, 92(92):273–292, 1992.

Vann McGee. Logical operations. *Journal of Philosophical Logic*, 25:567–580, 1996.

Andrzej Mostowski. On a generalization of quantifiers. *Fundamenta Mathematicae*, 44:12–36, 1957.

Gila Sher. *The Bounds of Logic: A Generalized Viewpoint*. MIT Press, 1991.

Gila Sher. Did Tarski commit 'Tarski's Fallacy'? *Journal of Symbolic Logic*, 61:653–686, 1996.

Alfred Tarski. Einige methodologische Untersuchungen über die Definierbarkeit der Begriffe. *Erkenntnis*, 5:80–100, 1935. quoted from translation with additions "Some methodological investigations on the definability of concepts" in (Tarski 1983, 296–319).

Alfred Tarski. *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*. Hackett Publishing, Indianapolis, second edition, 1983. John Corcoran (ed.).

Alfred Tarski. *Collected papers*, volume 1. Birkhäuser, Basel, 1986a.

Alfred Tarski. What are logical notions? *History and Philosophy of Logic*, 7:143–154, 1986b. edited by John Corcoran.

Timothy Williamson. Existence and contingency. *Aristotelian Society*, sup. vol. 73:227–234, 1999. Reprinted with printer's errors corrected in Proceedings of the Aristotelian Society 100 (2000): 321–343 (117–139 in unbound version).