POSSIBLE WORLDS

Content

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It allows one to find a common cause for many paradoxes.

It establishes a bridge to modal metaphysics.

It sheds a light on classical questions such as ante rem/in rebus conceptions of properties.

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Possible worlds for sentential operators

The language of (sentententential operator) modal logic treats the modal operator just like ¬.

If φ is a formula, so is $\underline{\Box}\varphi$, whereas for the modal *predicate* $\Box$ we can write at best $\Box\overline{\varphi}$ or $\Box x$.

Consequently, we can define the logical complexity of a formula in the lanugage of (sentententential operator) modal logic ion the usual way.

But we cannot define the logical complexity for formulæ with a modal predicate in the same way.

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But we cannot define the logical complexity for formulæ with a modal predicate in the same way.

A model for a language with a modal operator usually specifies a non-empty set W of worlds, an accessibility relation R on W, and an interpretation V that assign an interpretation to the non-logical vocabulary. For a standard first-order language, V gives applied to a world $w \in W$ a domain and to a world and a n-ary predicate P symbol an n-ary relation, and so on.

The truth of $\Box A$ at a world is then defined by induction on the complexity of A simultaneously for all worlds.

This is possible because formulæ in modal logic are wellfounded.

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Fundamentals

For pws semantics I make further assumption about the language $\mathcal{L}$.

$\mathcal{L}$ is the minimal language plus a single sentence symbol p. This the lanugage $\mathcal{L}_p$.

Of course, we would like to have more 'contingent' vocabulary. Here I keep things simple.

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$\mathcal{L}$ is the minimal language plus a single sentence symbol p. This the lanugage $\mathcal{L}_p$.

Of course, we would like to have more 'contingent' vocabulary. Here I keep things simple.

The standard model $\mathbb{E}$ of $\mathcal{L}_{qc}$ has as its domain the set of all expressions of the full language $\mathcal{L}$ and interprets the vocabulary such as the quotation constants and $\frown$ in the expected way.

A standard model for $\mathcal{L}_p$ is of the form $\langle \mathbb{E}, V, S \rangle$, where V assigns a truth value *true* or *false* to p and S is the extension of $\Box$, that is, we have the following:

$$\langle \mathbb{E}, V, S \rangle \models \Box \overline{e} \quad \text{iff} \quad e \in S.$$

The notion of a frame is exactly the same as in operator modal logic:

DEFINITION

A *frame* is an ordered pair $\langle W, R \rangle$ where W is non-empty and R is a binary relation on W.

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Definition

A *PW-model* is a quadruple $\langle W, R, V, B\rangle$ such that $\langle W, R\rangle$ is a frame, V is a valuation for $\langle W, R\rangle$, and B is a $\Box$-interpretation for $\langle W, R\rangle$ satisfying the following condition, where $\mathcal{L}_p$ is the set of all $\mathcal{L}_p$-sentences:

$$B(w) = \left\{ \varphi \in \mathcal{L}_p\colon \text{ for all } u \in W\text{: if } wRu \text{ then } \left\langle \mathbb{E}, V(u), B(u)\right\rangle \vDash \varphi \right\}.$$

$\langle \mathbb{E}, V(u), B(u) \rangle \vDash \varphi$ means that the sentence φ is true in the standard model $\langle \mathbb{E}, V(u), B(u) \rangle$ in the usual sense of first-order predicate logic; and the expression $\langle \mathbb{E}, V(u), B(u) \rangle \vDash \varphi$ can be read as 'φ is true at world u in the PW-model $\langle W, R, V, B \rangle$'.

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Lemma (normality)

Suppose $\langle W, R, V, B\rangle$ is a PW*-model, $w \in W$, and φ, ψ sentences of $\mathcal{L}_p$. Then the following hold:*

(i) *If $\left\langle \mathbb{E}, V(u), B(u)\right\rangle \vDash \varphi$ for all $u \in W$, then $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \vDash \Box\overline{\varphi}$.*

(ii) $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \vDash \Box\overline{\varphi \to \psi} \to (\Box\overline{\varphi} \to \Box\overline{\psi})$.

LEMMA

(i) *If a frame $\langle W, R\rangle$ is transitive and $\langle W, R, V, B\rangle$ a* PW*-model on that frame, we have for all sentences φ in $\mathcal{L}_p$ and worlds $w \in W$:*

$$\left\langle \mathbb{E}, V(w), B(w) \right\rangle \vDash \Box\overline{\varphi} \to \Box\overline{\Box\overline{\varphi}}.$$

(ii) *If a frame $\langle W, R\rangle$ is reflexive and $\langle W, R, V, B\rangle$ a* PW*-model on that frame, we have for all sentences φ in $\mathcal{L}_p$ and worlds $w \in W$:*

$$\left\langle \mathbb{E}, V(w), B(w) \right\rangle \vDash \Box\overline{\varphi} \to \varphi.$$

Definition

A frame $\langle W, R\rangle$ *admits a* PW*-model on every valuation* iff for every valuation V on $\langle W, R\rangle$ there is a B such that $\langle W, R, V, B\rangle$ is a PW-model. A frame *admits a* PW*-model* iff the frame admits a PW-model on some valuation, that is, iff there is a valuation V and a $\Box$-interpretation B such that $\langle W, R, V, B\rangle$ is a PW-model.

STRONG CHARACTERIZAITON PROBLEM

Which frames admit a PW-model on every valuation?

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Strong characterizaiton problem

Which frames admit a PW-model on every valuation?

Negative results

There is only one world, say, w, and this world can see itself. Thus we have $W_1 = \{w\}$ and $R_1 = \{\langle w, w\rangle\}$.

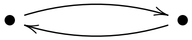
$$\begin{tikzcd} w \arrow[loop above] \end{tikzcd}$$

Theorem (liar paradox)

The above frame $\langle W_1, R_1\rangle$ does not admit a valuation.

Example (Montague)

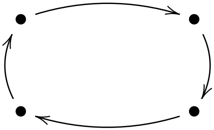
If $\langle W, R\rangle$ admits a valuation, then $\langle W, R\rangle$ is not reflexive.



Example

The frame ‘two worlds see each another’ displayed above does not admit a valuation.

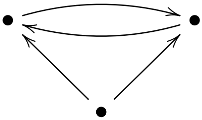
The following frame does not admit a PW-model.



EXAMPLE

The frame 'one world sees itself and one other world' does not admit a valuation.

For the proof the fixed point $\gamma \leftrightarrow (\Box\overline{\gamma} \to \Box\overline{\neg\gamma})$ can be employed.



EXAMPLE

The above frame 'one world sees two worlds that see each another' does not admit a valuation.

One can show this by using the fixed point $\gamma \leftrightarrow \neg\Box\overline{\Box\overline{\gamma}} \wedge \neg\Box\overline{\gamma}$

Example (McGee's paradox)

The frame $\langle \omega, \mathrm{Pre} \rangle$ does not admit a PW-model. Here ω is the set of all natural numbers and Pre is the successor relation. Hence every world n sees $n+1$ but no other world.

The frame $\langle \omega, \mathrm{Pre} \rangle$ can be displayed by the following diagram:

$$\begin{tikzcd}
0 \arrow[d] \\
1 \arrow[d] \\
2 \arrow[d] \\
\vdots
\end{tikzcd}$$

Theorem (Yablo–Visser paradox)

The frame $\langle \omega, < \rangle$ does not admit a PW*-model. Here $<$ is the usual 'smaller than' relation on the natural numbers:*

The frame $\langle \omega, < \rangle$ can be displayed by the following diagram:

$$\begin{tikzcd}
0 \arrow[d] \arrow[dd, bend left] \arrow[ddd, bend right] \\
1 \arrow[d] \arrow[dd, bend right] \\
2 \arrow[d] \arrow[d, bend left] \\
\vdots
\end{tikzcd}$$

Positive results

$$
\begin{tikzcd}
\vdots \arrow[d] \\
2 \arrow[d] \\
1 \arrow[d] \\
0
\end{tikzcd}
$$

By Suc we denote the successor relation $\{\langle k, n\rangle : \ k = n + 1\}$ on the set ω of natural numbers.

Example

The frame $\langle \omega, \mathrm{Suc}\rangle$ admits a PW-model on every valuation.

Obviously we can stop at any point.

What about the liar sentence?

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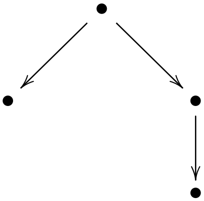
$$\begin{gathered}\left\langle \mathbb{E}, V(0), B(0) \right\rangle \vDash \neg\gamma, \\ \left\langle \mathbb{E}, V(1), B(1) \right\rangle \vDash \gamma, \\ \left\langle \mathbb{E}, V(2), B(2) \right\rangle \vDash \neg\gamma, \\ \left\langle \mathbb{E}, V(3), B(3) \right\rangle \vDash \gamma, \\ \vdots \end{gathered}$$



Example

The frame above admits a PW-model on every valuation.

We can prove this for any tree of this kind.

Definition

A frame $\langle W, R\rangle$ is converse wellfounded (or Noetherian) iff for every non-empty $M \subseteq W$ there is a $w \in M$ that is R-maximal in M.

Lemma

Every converse wellfounded frame $\langle W, R\rangle$ admits a PW*-model on every valuation.*

Proof.

Define B by recursion on the rank of w in R^{-1}:

$$B(w) := \{ \varphi \in \mathcal{L}_p : \forall u (wRu \to \langle \mathbb{E}, V(u), B(u)\rangle \models \varphi) \}$$

$\square$

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$\square$

The Strong Characterization Problem

Theorem (Strong Characterization theorem)

A frame admits a PW-model on every valuation iff it is converse wellfounded.

The lemma above yields the right-to-left direction.

The proof for the other direction proceeds via Löb's theorem.

THEOREM (STRONG CHARACTERIZATION THEOREM)

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The proof for the other direction proceeds via Löb's theorem.

The accessibility relation need not be transitive. We can define its transitive closure:

$$\mathsf{E} \vdash \forall n \forall \mathsf{z} \forall \mathsf{y} \Big(\gamma(n, \mathsf{z}, \mathsf{y}) \;\leftrightarrow$$
$$\exists k \left(n = k \frown \underline{1} \;\wedge\; \mathsf{z} = \overline{\forall \mathsf{z} \left(\gamma(\underset{\bullet}{k}, \mathsf{z}, \underset{\bullet}{\mathsf{y}}) \to \Box \mathsf{z} \right)} \right) \vee \left(n = \underline{0} \;\wedge\; \mathsf{z} = \mathsf{y} \right) \Big)$$

$$\Box^0 \overline{\varphi} \;:=\; \Box \overline{\varphi},$$
$$\Box^{n+1} \overline{\varphi} \;:=\; \Box \overline{\Box^n \overline{\varphi}}$$

Lemma

For every PW*-model* $\langle W, R, V, B \rangle$*, every* $w \in W$*, and all* $n \in \mathbb{N}$*,*

$$\Big\langle \mathbb{E}, V(w), B(w) \Big\rangle \vDash \forall \mathsf{z} \Big(\gamma(\underline{n}, \mathsf{z}, \overline{\varphi}) \to \Box \mathsf{z} \Big) \leftrightarrow \Box^n \overline{\varphi}$$

Then we define $\Box^*$ as follows:

$$\Box^* \mathsf{y} \;:=\; \forall n \forall \mathsf{z} \Big(\gamma(n, \mathsf{z}, \mathsf{y}) \to \Box \mathsf{z} \Big).$$

Lemma

For all φ in $\mathcal{L}_p^{\mathrm{S}}$, all PW*-models $\langle W, R, V, B\rangle$, and all $w \in W$,*

$$\left\langle \mathbb{E}, V(w), B(w) \right\rangle \vDash \Box^* \overline{\varphi} \quad \textit{iff} \quad \textit{for all } v \textit{ with } wR^*v\textit{:}\ \left\langle \mathbb{E}, V(v), B(v) \right\rangle \vDash \varphi.$$

Lemma

For all $\mathcal{L}_P$-sentences φ and ψ and PW*-models $\langle W, R, V, B\rangle$ the following hold:*

(i) *If* $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \models \varphi$ *for all* $w \in W$, *then* $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \models \Box^* \overline{\varphi}$.

(ii) $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \models \Box^* \overline{\varphi \to \psi} \to (\Box^* \overline{\varphi} \to \Box^* \overline{\psi})$.

(iii) $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \models \Box^* \overline{\varphi} \to \Box^* \overline{\Box^* \overline{\varphi}}$.

(iv) $\left\langle \mathbb{E}, V(w), B(w)\right\rangle \models \Box^* \overline{\Box^* \overline{\varphi} \to \varphi} \to \Box^* \overline{\varphi}$.

Lemma

The transitive closure R^* of the accessibility relation R of any frame that admits a PW*-model on every valuation is converse well-founded.*

Assume that $\langle W, R^* \rangle$ is converse ill-founded. Then there is a non-empty set $M \subseteq W$ without an R^*-maximal element. Define a valuation V as follows:

$$V(w)(p) := \begin{cases} \text{true}, & \text{if } w \notin M, \\ \text{false}, & \text{if } w \in M. \end{cases}$$

Lemma

A frame $\langle W, R \rangle$ is converse well-founded iff its transitive closure $\langle W, R^* \rangle$ is converse well-founded.

Theorem (Strong Characterization theorem)

A frame admits a PW*-model on every valuation iff it is converse wellfounded.*

DEEP INSIGHT

Löb's theorem is the mother of all paradoxes.

– at least in settings where we have pws.