Quantum field theory for condensed matter physics

Aims and prerequisites:

This course aims to give a brief introduction to quantum field theory for first-year graduate students in experimental condensed matter physics. The course assumes undergraduate quantum mechanics and undergraduate condensed matter physics.

Motivation:

Why study this subject? First, quantum field theory provides a beautiful and elegant description of the Universe in which we live. Second, the ideas of quantum field theory are foundational for condensed matter physics. Third, even if you are an experimentalist, you will read papers by theorists who invoke ideas from quantum field theory. This course is too brief to give you professional proficiency in the subject, but hopefully it will convey some of the key ideas and provide a springboard for further study.

There are many books on quantum field theory, though all written by theorists. My favourite ones include Peskin and Schroder, Ryder and Zee.

Synopsis:

1. Creation and annihilation operators and the harmonic oscillator
2. Coherent states
3. Lagrangians
4. The Hubbard model
5. Quantum fields
6. How to write down a theory
7. Symmetry and transformations
8. Broken symmetry

Exercises:

1. For the one-dimensional harmonic oscillator, show that the creation and annihilation operators defined in the lectures satisfy $[\hat{a},\hat{a}]=0$, $[\hat{a}^\dagger,\hat{a}^\dagger]=0$, $[\hat{a},\hat{a}^\dagger]=1$ and $\hat{H}=\hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})$.

2. Consider a perturbation to the one-dimensional harmonic oscillator Hamiltonian of the form $\beta\hat{x}^3+\gamma\hat{x}^4$ where β and γ are small. By writing the perturbation in terms of creation and annihilation operators of the original Hamiltonian, show that the first-order shift in the ground-state energy of the system, due to these anharmonic parts, is given by $\Delta E = \frac{3}{4}\gamma\left(\frac{\hbar}{m\omega}\right)^2$.

3. (a) Show that the transformation $\hat{b} = u\hat{a} + v\hat{a}^\dagger$ and $\hat{b}^\dagger = u\hat{a}^\dagger + v\hat{a}$ (with u and v real), preserves the commutation relations, as long as $u^2v^2 = 1$.

   (b) Using the results of (a), diagonalize the Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \frac{\Delta}{2}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}\right) \tag{1}$$

   by transforming it into the form $\hat{H} = \hbar\epsilon(\hat{b}^\dagger\hat{b} + \frac{1}{2})$ and find ϵ. This is an example of a Bogoliubov transformation and is a useful trick to diagonalize a Hamiltonian. *Hint: If you have a problem with the algebra, see J.F. Annett, Superconductivity, Superfluids and Condensates for some help.*

4. The nearest neighbour Hubbard model Hamiltonian may be written

$$\hat{H} = \sum_{ij\sigma}(-t_{ij})\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma} + U\sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}, \tag{2}$$

   where the first sum is over unique nearest neighbours. Consider a system with two possible sites for electrons.

   (a) Put a single electron in the system. Using a basis $|\uparrow, 0\rangle$ and $0, \uparrow\rangle$, show that the Hamitonian is given by

$$\hat{H} = \begin{pmatrix} 0 & -t \\ -t & 0 \end{pmatrix}. \tag{3}$$

   Find the energy eigenvalues and eigenstates.

   (b) Now put a second electron into the system with opposite spin to the first. Now using a basis such that a general state can be written as $|\psi\rangle = a|\uparrow\downarrow, 0\rangle + b|\uparrow, \downarrow\rangle + c|\downarrow, \uparrow\rangle + d|0, \uparrow\downarrow\rangle$, show that in this basis the Hubbard Hamiltonian is

$$\hat{H} = \begin{pmatrix} U & -t & -t & 0 \\ -t & 0 & 0 & -t \\ -t & 0 & 0 & -t \\ 0 & -t & -t & U \end{pmatrix}. \tag{4}$$

   Diagonalize this to obtain the eigenstates and energy eigenvalues. *Hint: There's no shame in using a computer if you like!*

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