Testing the effectiveness of unconventional monetary policy in Japan and the United States*

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Abstract

Unconventional monetary policy (UMP) may make the effective lower bound (ELB) on the short-term interest rate irrelevant. We develop an empirical test of this 'irrelevance hypothesis,' based on a simple idea that under the hypothesis, the short rate can be excluded in any empirical model that accounts for alternative measures of monetary policy. We develop a theoretical model that underpins this hypothesis, and test it empirically for Japan and the United States using a structural vector autoregressive model with the ELB. For each country, we firmly reject the hypothesis but find that UMP has had strong delayed effects.

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1 Introduction

Adjustments in the overnight nominal interest rate have been the primary tool for the implementation of monetary policy since the early 1980s. In recent years, however, the short-term nominal interest rate reached an effective lower bound (ELB) in several countries, making the standard policy tool *de facto* ineffective. Two prominent examples are Japan that reached the ELB since the domestic financial crisis of 1997-1998, and the United States that reached the ELB in the aftermath of the global financial crisis of 2007-2008. The central banks in these countries countervailed the inapplicability of the standard policy tool by embarking on unconventional monetary policy (UMP) that involves central bank purchasing of government bonds, and use of forward guidance to signal future policy action.[1]

The effectiveness of UMP is a central issue for policy makers. One view is that the ELB restricts the effectiveness of monetary policy, thus representing an important constraint on what monetary policy can achieve, as theoretically argued by Eggertsson and Woodford (2003). An alternative view is that UMP can affect long-term interest rates so significantly that UMP is fully effective in circumventing the ELB constraint, as empirically argued by Swanson and Williams (2014) and Debortoli et al. (2019) using U.S. data. This latter view has been termed as the ELB 'irrelevance hypothesis.' The issue of the effectiveness of UMP has gained relevance since the ELB will likely be binding for an extended period in several countries as of 2021.

This paper studies the ELB irrelevance hypothesis both empirically and theoretically. It develops a novel empirical test for the irrelevance hypothesis, based on a simple idea that under the hypothesis, the short-term interest rate can be excluded from any empirical model that includes alternative measures of monetary policy like long-term interest rates. It then formalizes the hypothesis by developing a dynamic stochastic general equilibrium (DSGE) model with UMP. Our empirical results show that the hypothesis is strongly rejected both for Japan and for the U.S. Despite the rejection, the estimated impulse responses to a monetary policy shock indicate strong delayed effects of UMP in each country.

By definition, the irrelevance hypothesis implies that the dynamic responses and the volatilities of macroeconomic variables remain unchanged when the economy moves in and

[1]See Christensen and Rudebusch (2012) and Liu et al. (2019) for the U.S., and Ugai (2007), and Bank of Japan (2016) for Japan. Ueda (2012) provides a comparison of monetary policy between the U.S. and Japan.

out of ELB regimes, for otherwise the ELB is empirically relevant. Therefore, a direct appraisal of the irrelevance hypothesis is to test whether short-term interest rates, which are subject to the ELB, can be excluded from vector autoregressions (VARs) that include alternative measures of monetary policy that are not subject to the ELB. A promising candidate for such measures is long-term interest rates, as various types of UMP implemented in many counties have intended to affect the long rates. The exclusion of the short rate in VARs that include the long rates is a novel empirical test developed in this paper. For robustness, we also conduct a test using a measure of desired monetary policy stance proposed by Mavroeidis (2021). These tests can be performed using reduced-form VARs, without the need to identify the policy shocks. However, because the regime switching induced by the ELB is endogenous as is indeed the case in practice, we cannot resort to standard VAR methods. Instead, we use the methodology in Mavroeidis (2021) that allows the estimation of VARs with endogenous regime-switching induced by occasionally binding constraints.

To formalize the irrelevance hypothesis and underpin our empirical tests, we develope a theoretical model with an ELB constraint in which UMP can replicate the effect of conventional monetary policy by using longer-term interest rates. To the best of our knowledge, this is the first study that characterizes analytically the irrelevance hypothesis in a DSGE model. In our theoretical framework, UMP consists of (i) quantitative easing implemented by long-term government bond purchases that directly affect long-term government bond yields, and (ii) forward guidance (FG) under which the central bank commits to keeping short-term interest rates low in the future. Under the mild assumption that the central bank continues to use inflation and the output gap as key indicators to guide the policy stance during ELB periods, UMP entails wide degrees of effectiveness, including the ELB irrelevance in which UMP retains the same effectiveness as the conventional policy that adjusts the short-term interest rate as if there were no ELB. We show that under the irrelevance of the ELB, the log-linearized DSGE model can be written in terms of inflation, the output-gap, and the long-term interest rate, and it retains the same VAR representation for both ELB and non-ELB regimes.

We conduct the tests of the ELB irrelevance hypothesis on postwar data for Japan and the U.S. We consider several different VAR specifications, varying the lag order and the estimation sample (to account for structural change), and using alternative measures of the variables in the model. In all cases, the tests overwhelmingly and consistently reject the hypothesis that the ELB has been empirically irrelevant for both economies. Our conclusion is therefore fairly robust: the ELB does represent a constraint on what monetary policy can achieve in those economies.[2]

The rejection of the irrelevance hypothesis leaves open the question on the degree of effectiveness of UMP compared to the conventional policy when interest rates are above the ELB. To address this question, we identify the dynamic effects of conventional and unconventional policies by combining the identifying power of the ELB with additional sign restrictions on impulse responses to a monetary policy shock à la Uhlig (2005). The ELB enables partial identification of the impulses responses to a monetary policy shock, as shown in Mavroeidis (2021). This is because a change in the behaviour of the economy across ELB and non-ELB regimes is informative about the relative impact of conventional and unconventional policy. The identified set based only on the ELB turns out to be fairly wide, so we use the insights from our DSGE model to impose the following theoretically-congruous sign restrictions that were used in Debortoli et al. (2019): a negative monetary policy shock should have a nonnegative effect on inflation and output and a nonpositive effect on the policy rate over the first four quarters. These sign restrictions markedly sharpen the identified set of impulse responses.

We find that the effects of monetary policy on inflation and output on impact (i.e., within the quarter) declined when the economy entered an ELB regime: they dropped by more than 20% in the U.S., and more than 50% in Japan, relative to conventional policy. However, the cumulative effects of policy exhibited the opposite pattern one and two years ahead: they appear to have been stronger during ELB regimes relative to non-ELB regimes, except for the response of output gap in the U.S., which remained weaker. Therefore, UMP seems to have had a delayed but stronger effect than conventional policy on inflation in the U.S., and on both inflation and output in Japan. Thus, we conclusively reject the hypothesis that the ELB has been empirically irrelevant in both countries, and find that responses of inflation and output to UMP have been different across time and across countries.

Finally, both our theoretical and empirical models include a 'shadow rate' defined as the short-term interest rate that the central bank would set if there were no ELB. Thus defined, the shadow rate can be interpreted as an indicator of the desired monetary policy stance and

[2]This evidence corroborates Eggertsson and Woodford (2003) who claim that "the zero bound does represent an important constraint on what monetary stabilization policy can achieve", and is consistent with the findings in Gust et al. (2017) and Del Negro et al. (2017), who attribute an important role to the ELB for the decline in output during the financial crisis.

we provide estimates of it for Japan and the U.S. Our estimates of the shadow rate do not impose the assumption that the model used to obtain them is constant across regimes, and therefore they explicitly account for the empirical relevance of the ELB over the estimation periods.

Related literature Our analysis is closely related to two strands of research. The first strand of literature pertains to theoretical studies that investigate the transmission mechanism of unconventional monetary policy. Among those, regarding QE, our theoretical model is close in spirit to Andrés et al. (2004), Chen et al. (2012), Harrison (2012), Gertler and Karadi (2013), Liu et al. (2019), and Sudo and Tanaka (2020). These studies use heterogeneous preferences for assets of different maturities and limit arbitrage across assets to break the irrelevance of QE, as discussed in Eggertsson and Woodford (2003).[3] Regarding FG, our model follows Reifschneider and Williams (2000), and it considers this mechanism in a general equilibrium model that directly accounts for purchasing of long-term bonds. Our main contribution to this first strand of literature is to develop a simple model of UMP, which incorporates the shadow rate and provides theoretical underpinnings to our empirical analyses.

The second strand of literature pertains to empirical studies that assess the effectiveness of unconventional policy. It includes Swanson and Williams (2014) and Debortoli et al. (2019), who use SVARs to investigate the (ir)relevance of the ELB constraint by comparing impulse responses to shocks between normal times and ELB episodes. Differing from our SVAR, these studies do not include short-term interest rates in their SVARs. Another related study by Inoue and Rossi (2019) uses an SVAR with shocks to the entire yield curve and finds evidence that UMP has been effective in the U.S. Our empirical methodology is closely related to Hayashi and Koeda's (2019), who propose an SVAR model for Japan that includes short rates and takes into account the ELB, and our empirical model for Japan relies heavily on the insights from their empirical analysis. The main difference of our methodology from Hayashi and Koeda's (2019) is that ours uses a shadow rate to model UMP, which nests both QE and FG via a Reifschneider and Williams (2000) policy rule, while Hayashi and Koeda (2019) use excess reserves to model QE and an inflation exit condition to model FG. Our methodology provides a simpler framework to test the ELB irrelevance hypothesis and

[3] For other possible channels of QE, see Krishnamurthy and Vissing-Jorgensen (2011). See also Sims and Wu (2020) for a recent discussion on the theoretical frameworks to study UMP.

to compare the effectiveness of UMP relative to conventional policy.[4]

Finally, our empirical analysis uses the estimation methodology in Mavroeidis (2021), who also reports evidence against the irrelevance hypothesis for the U.S. using a three-equation VAR model. Relative to that paper, in addition to our novel theoretical DSGE model of UMP, we provide a new test of the irrelevance hypothesis, we study Japanese data and conduct several robustness analyses, we use sign restrictions motivated from our theoretical model to sharpen the identification of impulse responses, and we estimate the dynamic effects of UMP and the shadow rates in each country.

The structure of the paper is as follows. Section 2 introduces the econometric methodology that will be used in the empirical analysis. Section 3 provides theoretical underpinnings to the empirical analysis via a simple New Keynesian DSGE model with UMP. Section 4 describes the data. Section 5 reports the empirical results. Section 6 concludes. Appendices provide supporting material on the derivation of the DSGE model, data description, and additional empirical results.

2 Empirical model

In this section, we provide an empirical model that allows us to achieve two objectives. Our first objective is to provide formal statistical evidence on the so-called ELB irrelevance hypothesis. Our second objective is to obtain estimates of the effectiveness of UMP relative to conventional policy.

2.1 Censored and kinked SVAR

We will carry out our empirical analysis using an agnostic SVAR in which the short-term interest rate is subject to an ELB constraint. The econometric model that we will use is the censored and kinked SVAR (CKSVAR) developed by Mavroeidis (2021), described by the following equations:

$$Y_{1t} = \beta\left(\lambda Y_{2t}^{*} + (1-\lambda)\,Y_{2t}\right) + B_1 X_t + B_{12}^{*} X_{2t}^{*} + A_{11}^{-1}\varepsilon_{1t}, \tag{1}$$

$$Y_{2t}^{*} = -\alpha Y_{2t} + (1+\alpha)\left(\gamma Y_{1t} + B_2 X_t + B_{22}^{*} X_{2t}^{*} + A_{22}^{*-1}\varepsilon_{2t}\right), \tag{2}$$

$$Y_{2t} = \max\left\{Y_{2t}^{*}, b_t\right\}, \tag{3}$$

[4] Moreover, despite the apparent methodological differences, and the different samples, our estimates for Japan are consistent with those reported in Hayashi and Koeda (2019). See Section 5 for further discussion.

where $Y_t = (Y_{1t}', Y_{2t})'$ is a $k \times 1$ vector of endogenous variables, partitioned such that the $k-1$ variables Y_{1t} are unconstrained and the scalar Y_{2t} is constrained, X_t comprises exogenous and predetermined variables, including lags of Y_t, X_{2t}^* consists of lags of Y_{2t}^*, ε_t are i.i.d. structural shocks with identity covariance matrix, b_t is an observable lower bound, and $Y_{2t}^* < b_t$ is unobservable. The 'latent' variable or the shadow rate Y_{2t}^* in this model will represent the *desired* policy stance, as opposed to the *effective* policy stance, e.g., in Wu and Xia (2016), except in the special case $\alpha = 0$ and $\lambda = 1$. When $Y_{2t}^* < b_t$, the shadow rate represents UMP, such as QE or FG, which are not modelled explicitly. The DSGE model of Section 3 provides a theoretical justification for this interpretation.

Equation (2) nests the FG rule of Reifschneider and Williams (2000), and the parameter α has the same interpretation as in the theoretical model of Section 3: a large α implies that interest rates will stay at the ELB for longer. As we will discuss further in the next section, FG is also captured by the coefficients on the lags of the shadow rate in the policy rule, B_{22}^*.[5]

The parameter λ partially characterizes the effectiveness of UMP relative to conventional policy *on impact*. Specifically, from equation (1) we see that above the ELB (i.e., when $Y_{2t} = Y_{2t}^* > b_t$), the contemporaneous effect of a change in the short-term interest rate Y_{2t} by one unit on Y_{1t} is β, but the corresponding effect at the ELB, driven by a change in Y_{2t}^*, is $\lambda\beta$. When $\lambda = 1$, the two effects are equal, while $\lambda = 0$ corresponds to the case in which UMP has no contemporaneous effect on Y_{1t}.

However, the parameter λ does not suffice to pin down impulse responses to a monetary policy shock at the ELB. The intuition is straightforward. Consider the impulse response to a unit change in the monetary policy shock $A_{22}^{*-1}\varepsilon_{2t}$ ignoring nonlinearities.[6] The effect on Y_{1t} is $\beta/(1-\gamma\beta)$ above the ELB, and $\xi\beta/(1-\xi\gamma\beta)$ at the ELB, where:

$$\xi = \lambda(1+\alpha). \tag{4}$$

So, it is, in fact, ξ, not λ, that measures the effectiveness of an UMP shock – a shock to the shadow rate below the ELB. To see why, consider an illustrative example in which a UMP shock would have been only, say, 50% as effective if it had the same magnitude as a conventional MP shock ($\lambda = 0.5$), but the central bank retains the nominal interest rate at

[5]See also the discussion on interest rate smoothing, ρ_i, in the Taylor rule (12) in Section 3. Note that the dynamics of the policy rule in equation (2) are completely unrestricted, whereas the specification of the Taylor rule (12) excludes lags of Y_{1t}. Thus, the empirical analysis does not rely on any short-run exclusion restrictions for identification.

[6]The exact specification of the IRF is given by equation (16) with $h = 0$.

the ELB longer than would have been implied by the conventional Taylor rule, and that this can be represented with a value of $\alpha = 1$. This, in turn, would cause the effective UMP shock to be twice as large as the conventional shock $A_{22}^{*-1}\varepsilon_{2t}$. Then the observed impact of such a UMP shock will be of the same magnitude as the corresponding conventional policy shock.[7]

Our discussion about the interpretation of the parameter ξ concerned the relative effectiveness of UMP *on impact*. The *dynamic* effects of UMP on Y_{1t} are governed by the coefficients on the lags of the shadow rate B_{12}^{*}. For example, the case of completely ineffective UMP at all horizons can be represented by the joint restrictions $\lambda = 0$ and $B_{12}^{*} = 0$. A more restrictive assumption is that UMP has no effect on the conventional policy instrument Y_{2t} either, i.e., that any FG or QE is completely ineffective in changing the path of short-term interest rates as well. This can be represented by the particular case of $\lambda = \alpha = 0$, $B_{12}^{*} = 0$ and $B_{22}^{*} = 0$, which implies that the shadow rate completely drops out of the right-hand side of equations (1)-(3). Mavroeidis (2021) calls this particular case a *kinked* SVAR (KSVAR), and shows that the absence of latent regressors in the likelihood function makes the KSVAR much easier to estimate than the CKSVAR, which will be helpful in our analysis.

2.2 ELB irrelevance hypothesis

A central implication of the ELB irrelevance hypothesis is that the dynamics of the economy are independent from whether policy rates are at the ELB or not. We will exploit this implication to formulate two testable hypotheses in the context of an SVAR.

The first testable hypothesis is based on the work of Swanson and Williams (2014) and Debortoli et al. (2019), who perform causal inference using an SVAR that includes long rates but excludes short rates. Specifically, we will test the hypothesis that short rates can be excluded from equation (1) in the vector of observables Y_{1t} if long rates are included in Y_{1t}. We show that the DSGE model provides a theoretical underpinning to this testable implication (see Proposition 1 in Section 3). Note that we test this hypothesis without any particular identification assumptions on the SVAR under the null hypothesis, i.e., we can simply test that short rates can be excluded once long-rates are included in the SVAR. In other words, any causal inference can be based on a VAR in Y_{1t} only, as in Swanson and Williams (2014) and Debortoli et al. (2019).

[7] See Mavroeidis (2021) for further discussion.

The second testable hypothesis is that a VAR for $(Y_{1t}', Y_{2t}^*)'$ is sufficient to capture the stance of monetary policy and there are no changes in the dynamics, correlations, or volatilities of the data across regimes. This is a special case of the general CKSVAR that requires $\xi = 1$ and some specific restrictions on the coefficients of the lags in the VAR, and is referred to as a purely *censored* SVAR (CSVAR) in Mavroeidis (2021). The requisite parametric restrictions can be imposed directly on the reduced form, see equation (15) in Section 5 below.

2.3 Identification

The methodology for the identification and estimation of the CKSVAR is developed in Mavroeidis (2021), where it is shown that the model is generally under-identified, but that the parameter ξ, defined in equation (4), as well as the impulse responses to the monetary policy shock ε_{2t}, are partially identified in general. To gain intuition for this result, it is useful to consider the solution (reduced-form) of equations (1)-(3) for Y_{1t} and Y_{2t}:

$$Y_{1t} = C_{11}X_{1t} + C_{12}X_{2t} + C_{12}^*X_{2t}^* + u_{1t} - \widetilde{\beta} D_t \left(C_2 X_t + C_{22}^* X_{2t}^* + u_{2t} - b_t\right) \quad (5)$$

$$Y_{2t} = \max\left\{C_{21}X_{1t} + C_{22}X_{2t} + C_{22}^*X_{2t}^* + u_{2t}, b_t\right\} \quad (6)$$

where $D_t := 1_{\{Y_{2t}=b_t\}}$ is the indicator of the ELB regime, $X_t = (X_{1t}', X_{2t}')'$, X_{2t} consists of the lags of Y_{2t} in X_t, the matrices C_{11}, C_{12}, C_{12}^*, $C_2 = (C_{21}, C_{22})$, and C_{22}^* are reduced-form coefficients, $u_t = (u_{1t}', u_{2t}')'$ are reduced-form errors, and $\Omega = var\left(u_t\right)$. The reduced-form equation (5) is an 'incidentally kinked' regression, whose coefficients and variance change across regimes. The coefficient of the kink $\widetilde{\beta}$ is identified, together with the remaining reduced-form parameters. In other words, we can infer from the data whether the slope coefficients and the variance of Y_{1t} change across regimes by testing whether $\widetilde{\beta} = 0$. However, the parameter $\widetilde{\beta}$ does not have a structural interpretation and relates to the underlying structural parameters through the equations:

$$\widetilde{\beta} = (1 - \xi)\left(I - \xi\beta\gamma\right)^{-1}\beta, \quad (7)$$

$$\gamma = \left(\Omega_{12}' - \Omega_{22}\beta'\right)\left(\Omega_{11} - \Omega_{12}\beta'\right)^{-1}. \quad (8)$$

Mavroeidis (2021) shows that the structural parameters ξ, β and γ are generally only partially identified, in the sense that there is a set of values of ξ, β and γ that correspond to any given value of the reduced form parameters $\widetilde{\beta}, \Omega$ – this is the set of solutions of equations (7) and

(8).[8] Therefore, the impulse responses to a monetary policy shock are set-identified. In our empirical analysis below, we will use sign restrictions on the impulse responses to further sharpen the identified set.

3 A theoretical model of UMP

In this section, we present a simple theoretical model of UMP. The model provides theoretical underpinnings to the empirical model – the CKSVAR – and it lays out important channels for the propagation mechanism of UMP.

In the following, Subsection 3.1 presents the model's central equations, Subsection 3.2 establishes how the model can be nested within the empirical model presented in Section 2, and Subsection 3.3 simulates the model to study UMP, the ELB irrelevance hypothesis, and impulse responses to a monetary policy shock. The details of model description, equation derivations, parameterization, and model simulations are reported in Appendix A.

3.1 Central equations

The model is a New Keynesian model with QE and FG under the ELB. The economy consists of households, firms, and a central bank. The firm sector is standard as in a typical New Keynesian model. The household sector comprises two types of households. Constrained households purchase long-term government bonds only, but unconstrained households can trade both short- and long-term government bonds subject to a trading cost. The trading cost captures bond market segmentation, as in the preferred habitat theory originally proposed by Modigliani and Sutch (1966), and it introduces imperfect substitutability between long- and short-term government bonds that generates a spread between these bonds' yields.[9]

The central bank follows a standard Taylor rule when the interest rate is above the ELB, but it may undertake FG and QE under the ELB. In particular, the central bank conducts FG using a monetary policy rule that commits itself to maintain the interest rate lower than the level implied by the Taylor rule, as in Reifschneider and Williams (2000). The central bank conducts QE by purchasing long-term government bonds using a shadow rate – the interest rate the central bank would set if there were no ELB – as policy guidance. The

[8] This set reduces to a point in the special case $\xi = 0$, when $\beta = \widetilde{\beta}$.

[9] The preferred habit model is the predominant modelling framework to study UMP. See among others Chen et al. (2012), Liu et al. (2019), and Sims and Wu (2020).

conduct of QE, which follows a policy rule as a function of the shadow rate, distinguishes this model from others, and plays a crucial role in providing theoretical underpinnings to our empirical model.

The model's equilibrium conditions are log-linearized and arranged into three blocks of equations: an Euler equation, a Phillips curve, and a monetary policy rule. Let $\hat{y}_t$, $\hat{\pi}_t$, and $\hat{i}_t$ denote output, inflation, and the short-term interest rate in period t in terms of a deviation from steady state.[10] Similarly, let $\hat{i}^*_t$ and $\hat{i}^{\text{Taylor}}_t$ denote the shadow rate and the Taylor-rule interest rate. As derived in Appendix A.2, the system of equations for the five variables is given by:

$$\hat{y}_t = E_t\hat{y}_{t+1} - \frac{1}{\sigma}\left((1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}^*_t - E_t\hat{\pi}_{t+1}\right) - \chi_b z^b_t, \tag{9}$$

$$\hat{\pi}_t = \delta E_t\hat{\pi}_{t+1} + \kappa\hat{y}_t - \chi_a z^a_t, \tag{10}$$

$$\hat{i}^*_t = -\alpha\hat{i}_t + (1+\alpha)\hat{i}^{\text{Taylor}}_t, \tag{11}$$

$$\hat{i}^{\text{Taylor}}_t = \rho_i\left((1-\lambda^*)\hat{i}_{t-1} + \lambda^*\hat{i}^*_{t-1}\right) + (1-\rho_i)\left(r_\pi\hat{\pi}_t + r_y\hat{y}_t\right) + \epsilon^i_t, \tag{12}$$

$$\hat{i}_t = \max\left\{\hat{i}^*_t, b\right\}, \tag{13}$$

where z^a_t and z^b_t are a productivity shock and a demand (preference) shock, respectively, both of which follow AR(1) processes, ϵ^i_t is a monetary policy shock, and $b = -i/(1+i)$ is the ELB of the interest rate in deviation from the steady state.

Equation (9) is the Euler equation, enriched to incorporate QE – purchases of long-term bonds. An increase in QE lowers the long-term interest rate by compressing the spread, stimulates consumption by restricted households, as these households hold long-term bonds only, and eventually increases output. The long rate and QE do not directly appear in the system including equation (9), because the long rate depends on the spread, which in turn depends on QE, and QE is conducted using the shadow rate $\hat{i}^*_t$, which encapsulates the desired stance of monetary policy, as policy guidance. In equation (9), both the long rate and QE are substituted for the shadow rate $\hat{i}^*_t$, and the associated parameter λ^* summarizes the effectiveness of QE. Indeed, as shown in Appendix A.2, the parameter λ^* accounts for the central bank's reaction of long-term bond acquisition in response to a change in $\hat{i}^*_t$, the elasticity of the spread with respect to such acquisition, and the degree of bond market segmentation. In the special case of $\lambda^* = 0$ (e.g., zero elasticity of the spread), QE has

[10] For the interest rate, a deviation from steady state is defined in terms of the gross interest rate. That is, $\hat{i}_t = (i_t - i)/(1+i)$, where i is the short-term net interest rate in steady state. Hence, under the assumption of $i_t \geq 0$, the ELB of $\hat{i}_t$ can be written as $-i/(1+i)$.

no effect on output, and equation (9) nests the standard Euler equation that abstracts from the shadow rate. In the more general case of $\lambda^* > 0$, the "effective" short-term rate, $(1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*$, can be below the ELB despite the short-term interest rate $\hat{i}_t$ is constrained by the ELB. In particular, the special case of $\lambda^* = 1$ represents the irrelevance of the ELB in which the central bank sets the *effective* interest rate at the desired interest rate of $\hat{i}_t^*$, and monetary policy achieves the same effectiveness as when the interest rate is above the ELB.

Equation (10) is a standard New Keynesian Phillips curve that relates the current inflation to the expected inflation at time $t+1$, the output gap, and the productivity shock. Equations (11)-(13) describe the monetary policy rule. Equation (13) corresponds to the ELB constraint: the central bank sets the interest rate $\hat{i}_t$ at the shadow rate $\hat{i}_t^*$ subject to the ELB represented by the parameter b. Equation (12) is a standard Taylor rule, enriched with interest-rate smoothing such that the Taylor rate $\hat{i}_t^{\text{Taylor}}$ depends on the lagged *effective* interest rate with the smoothing parameter $0 < \rho_i < 1$. Our modelling choice allows the parameter λ^* to control the effectiveness of FG as well. If $\lambda^* = 0$, the lagged *effective* interest rate collapses to the short rate $\hat{i}_t$. Under the ELB, the lagged short rate has no additional downward pressure on $\hat{i}_t^{\text{Taylor}}$ because $\hat{i}_t$ remains at the ELB. When $\lambda^* > 0$, however, an additional downward pressure on $\hat{i}_t^{\text{Taylor}}$ is exerted from $\lambda^*\hat{i}_{t-1}$, which is analogous to the FG in Debortoli et al. (2019). In this sense, the parameter λ^* encapsulates the impact of UMP – a combination of both QE and FG – that mirrors the role of the parameter λ in the empirical model presented in Section 2. The parameter $\alpha \geq 0$ in equation (11) provides an additional strength to FG, by maintaining the shadow rate lower than the rate set by the Taylor rule, in the spirit of Reifschneider and Williams (2000), and consistent with the empirical model presented in Section 2. Thus, when the central bank purchases long-term bonds and $\lambda^* > 0$, the parameter $\alpha > 0$ represents the effect of FG.

3.2 Mapping the theoretical model to the empirical model

In general, the theoretical model under rational expectations cannot be perfectly nested within the empirical model presented in Section 2. This is because such a theoretical model cannot be solved analytically, while the empirical model has a piecewise linear solution, as shown in equation (5). However, under specific cases, which we describe below, the solution to the theoretical model becomes identical to the representation of the empirical model. In what follows, we formulate two specific cases that develop microfoundations to the empirical model and provide the theoretical underpinning to the empirical tests.

Before discussing the two specific cases, it is useful to note that the theoretical framework prevents a separate identification of the parameters λ^* and α, while it allows the identification of the parameter $\xi^* = \lambda^*(1+\alpha)$, as in the empirical model. More formally:

Lemma 1 *For any $\lambda^* \neq \lambda^{*\prime}$ and $\alpha \neq \alpha'$ that satisfy $\xi^* = \lambda^*(1+\alpha) = \lambda^{*\prime}(1+\alpha') = \xi^{*\prime}$, the model with λ^* and α is observationally equivalent to the model with $\lambda^{*\prime}$ and α'.*

The proof is straightforward. By using equation (11) to substitute out $\hat{i}_t^*$, the effective interest rate, $(1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*$, is replaced with $(1-\xi^*)\hat{i}_t + \xi^*\hat{i}_t^{\text{Taylor}}$. The parameter λ^* does not appear anywhere in the model except in $\xi^* = \lambda^*(1+\alpha)$. The parameter α only appears in equation (13) as $\hat{i}_t = \max\{\hat{i}_t^{\text{Taylor}} - \alpha(\hat{i}_t - \hat{i}_t^{\text{Taylor}}), b\}$. But this equation is observationally equivalent to $\hat{i}_t = \max\{\hat{i}_t^{\text{Taylor}}, b\}$ because only $\hat{i}_t$ is observable. Hence, the theoretical model depends only on ξ^*.

Now we are in a position to establish the first case that provides theoretical underpinnings to our empirical analysis. Consider the case with UMP (i.e., the combination of λ^* and α) as effective as the conventional policy, such that $\xi^* = \lambda^*(1+\alpha) = 1$, and the ELB is irrelevant for policy. This case corresponds to the ELB irrelevance hypothesis. The next proposition shows that a solution to the theoretical model entails a VAR(1) representation.

Proposition 1 *Consider the theoretical model in equations (9)-(13) and assume $\xi^* = 1$. Then, (i) a solution for $[\hat{y}_t, \hat{\pi}_t, \hat{i}_t^*]$ has a VAR(1) representation, and (ii) a solution for $[\hat{y}_t, \hat{\pi}_t, \hat{\hat{R}}_{L,t}]$ has also a VAR(1) representation, where $\hat{\hat{R}}_{L,t}$ is a long-term government bond yield.*

Proof. Appendix A.4. ■

For an econometrician, the shadow rate is observable only when it is above the ELB. Hence, a solution for $[\hat{y}_t, \hat{\pi}_t, \hat{i}_t^*]$ has a VAR(1) representation but the shadow rate is censored at the ELB. Part (i) of the proposition shows that a Censored SVAR (CSVAR) – a special case of the CKSVAR – nests the theoretical model with $\xi^* = 1$. Part (ii) of the proposition establishes that the short rate (or the shadow rate) is redundant for describing the law of motions for inflation and output once the long rate is included in the system when UMP is as effective as the conventional policy. In the non-ELB regime, the expectation hypothesis with constant spread holds, and thereby the short rate can be substituted for the long rate. In the ELB regime with $\xi^* = 1$, the long rate, adjusted by UMP, moves exactly the same manner as in the non-ELB regime, giving rise to the VAR(1) representation with the long rate. These implications of the ELB irrelevance hypothesis are both testable, as discussed in Section 2. Proposition 1 provides a theoretical basis for our empirical tests.

The second case allows UMP to be less effective than conventional policy in the current period t (i.e., $\xi^* \leq 1$), and assumes that agents form expectations on the presumption that UMP will be as effective as conventional policy in the next period $t+1$ (i.e, $\xi^* = 1$). The rationale for this exercise is motivated by Aruoba et al. (2020), who show that a piecewise linear solution to a DSGE model with occasionally binding constraints can be interpreted as describing the behavior of boundedly rational agents. We take this interpretation and show that the theoretical model is nested within the empirical model in the special case in which agents understand that UMP may be partially effective, $\xi^* \leq 1$, in the current period t, but they expect UMP to be as effective as conventional policy from period $t+1$ onward, therefore forming expectations based on decision rules with $\xi^* = 1$.

Proposition 2 *Assume that in each period t the true value of $\xi^* \leq 1$ is known, and expectations are formed under the presumption of $\xi^* = 1$ from period $t+1$ onward. Then, the theoretical model given by equations (9)-(13) is nested within the empirical CKSVAR model given by equations (1)-(3).*

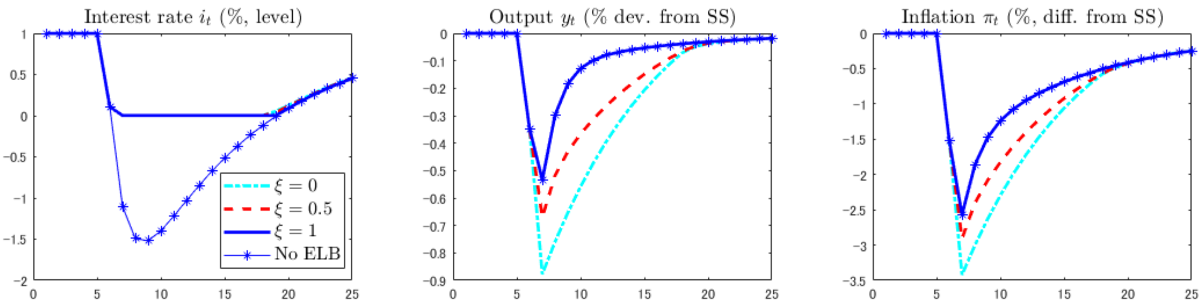
Proof. Appendix A.5 ■

Proposition 2 establishes that if agents expect UMP to be as effective as conventional policy in the future, the system of variables $[\hat{y}_t, \hat{\pi}_t, \hat{i}_t^*, \hat{i}_t]$ can be represented by the CKSVAR, despite ξ^* not being constrained to be unity in the current period. It is worth noting that the theoretical model under Proposition 2 is more restrictive than the empirical CKSVAR model. Hereafter, we use ξ instead of ξ^* for the theoretical model under Proposition 2 because in such a case the *impact* responses of the variables are identical between the theoretical and empirical models, and $\xi^* = \xi$.

3.3 Model simulations

In this section, we use the theoretical model under Proposition 2 to study the effect of UMP, the ELB irrelevance, and impulse responses to a monetary policy shock under the ELB. Our simple model embodies the transmission mechanisms of UMP at work with great transparency, and it is nested within the empirical model so that $\xi^* = \xi$. The calibrated

Figure 1: The effects of UMP



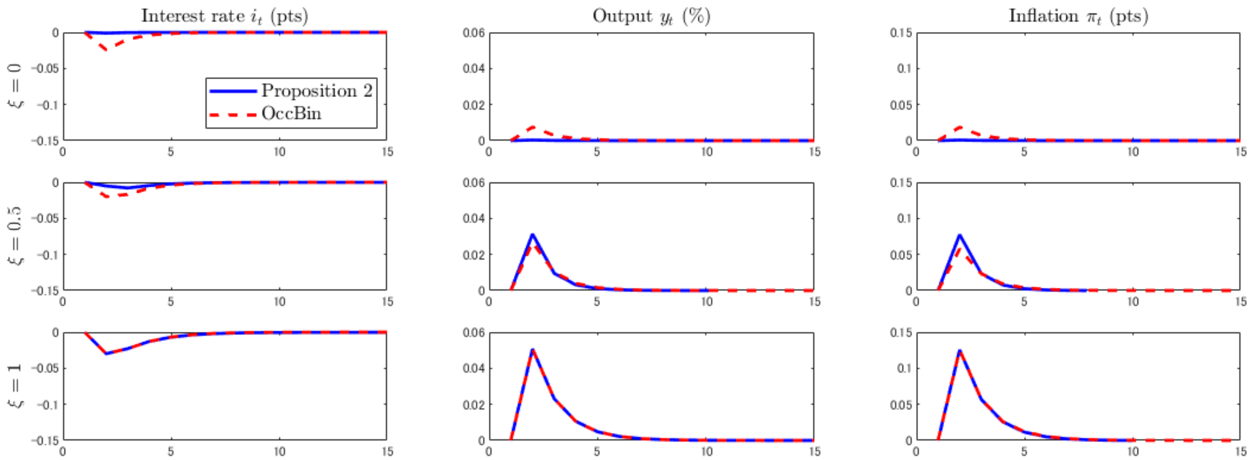
Note: The figure shows the dynamic path of the model under Proposition 2 where a severe demand shock hits the economy in periods $t = 6$ and $t = 7$. The dynamic path is computed by transforming the theoretical model under Proposition 2 into the reduced form equations (5) and (6) and calculating the response to the shocks.'No ELB' represents the model without the ELB, where the interest rate equation (13) is replaced by $i_t = i_t^*$. 'SS' denotes a steady state, 'dev.' denotes a deviation, and 'diff.' denotes a difference.

model is intended to illustrate the key theoretical mechanisms, and it is not designed to draw quantitative implications. Appendix A.3 reports the parameterization of the model.

No UMP. The dash-dotted line in Figure 1 shows simulated paths for the theoretical model under Proposition 2 for the case of no UMP ($\xi = 0$). The economy starts from the steady state and large negative demand shocks hit the system in periods $t = 6$ and $t = 7$. The succession of negative demand shocks brings the economy to the ELB and generates a severe recession by decreasing output and inflation sharply. At the ELB (dash-dotted line), the interest rate i_t cannot be lowered in response to the fall in inflation. This raises the real interest rate, decreases consumption and output, and puts further downward pressure on inflation through the Phillips curve (10). This negative feedback loop magnifies the falls in output and inflation compared to the hypothetical economy without the ELB (star-marked line).

UMP. UMP – the combination of QE and FG – can offset the negative impact of the ELB. When UMP is partially effective ($\xi = 0.5$; the dashed line), the magnitude of the falls in output and inflation are mitigated relative to the case without UMP ($\xi = 0$; the dash-dotted line). When UMP is fully effective ($\xi = 1$; the solid line), although the interest rate i_t remains at zero, output and inflation follow the same paths as in the case of no ELB (star-marked line), as shown in Figure 1. In response to a decrease in the shadow rate i_t^*, which is partly driven by FG, encapsulated by the parameter α, the central bank increases the purchase

Figure 2: Impulse responses to a monetary policy shock at the ELB



Note: 'Proposition 2' denotes the theoretical model under Proposition 2 and 'OccBin' denotes the model (equations (9)-(13)) solved by the algorithm developed by Guerrieri and Iacoviello (2015). For analyzing the impulse responses under the ELB, for each case of ξ, the initial condition is set as endogenous variables which are realized using OccBin in response to a severe negative demand shock.

of long-term government bonds and, by doing so, it lowers the long-term government bond yield by compressing its premium, which boosts consumption and output. When $\xi = 1$, UMP perfectly offsets the contractionary effect of the ELB on impact. The interest rate i_t disappears and becomes irrelevant to the dynamics of the system, and the economy evolves as if there were no ELB. In other words, the ELB becomes irrelevant for the dynamics of the economy when $\xi = 1$.

Impulse responses to a monetary policy shock. Figure 2 plots impulse responses to a 0.25 percentage points cut in the shadow rate under the ELB starting from period $t = 1$ for the theoretical model (equations (9)-(13)) solved under Proposition 2 (solid line) and the model solved by the OccBin algorithm (dashed line), developed by Guerrieri and Iacoviello (2015).[11] We report solutions of the model using the OccBin algorithm since it is a practical approach to solving DSGE model at the ELB (see Atkinson et al., 2019). Similar to our solution of the model under Proposition 2, the OccBin algorithm assumes that the non-ELB regime is absorbing and the interest rate remains positive once the economy exits the ELB regime, but unlike our solution it does not assume expectations of full effectiveness of UMP in the future.

[11]The impulse responses are computed by using the same method employed in reporting our empirical results. For the detail of the calculation, see Section 5.3.

Overall the responses of the interest rate, output, and inflation are similar between the model solution under Proposition 2 and the OccBin solution, as shown in Figure 2. The responses of the interest rate are muted because the economy starts from the ELB triggered by a severe demand shock in period $t = 1$. Without the ELB, the interest rate (left panels) would fall by about 0.15 percentage points (pts), reported in the figure as the lowest value on the y-axis. In the case of no UMP ($\xi = 0$; top panels), the responses of output (central panels) and inflation (right panels) are muted for both the model solution under Proposition 2 and the OccBin solution. Because the economy is at the ELB, the monetary policy shock in period $t = 2$ does not have significant effects on the economy without UMP. In the case of partial UMP ($\xi = 0.5$; middle panels), QE is activated in response to a decrease in the shadow rate triggered by the monetary policy shock, and output and inflation increase. In the case of fully effective UMP ($\xi = 1$; bottom panels), the 'irrelevance hypothesis' holds and the responses of output and inflation coincide with those under the hypothetical economy with no ELB under both solution methods.

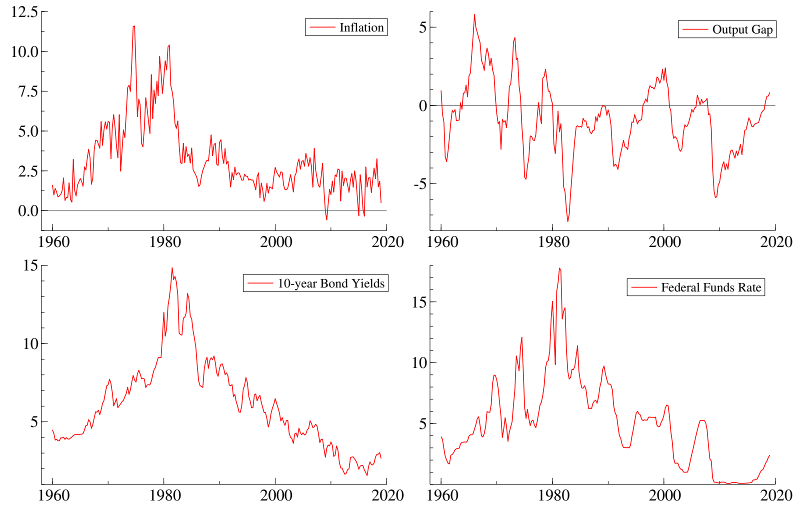
4 Data

Our empirical analysis focuses on the U.S. and Japan. We choose data series for the baseline specification of the SVAR model to maintain the closest specification as possible to related studies and thereby include representative series for inflation, output, and measures for short- and long-term yields.

For the U.S., we use quarterly data for inflation based on the GDP deflator, a measure of the output gap from the Congressional Budget Office, the short-term interest rate from the Federal Funds Rate, and the 10-year government bond yield from the 10-year Treasury constant maturity rate. Figure 3 plots these series. We also consider the different measures of money listed in Appendix B. The data are from the FRED database at the Federal Reserve Bank of St. Louis and the Center for Financial Stability databases. The estimation sample for the baseline specification is from 1960q1 to 2019q1.[12] We set the value of the effective lower bound on the Federal Funds Rate equal to 0.2, such that the short-term interest rate is at the ELB regime 11% of the time, which is consistent with Bernanke and Reinhart (2004) who suggest that the effective lower bound on nominal interest rates may be above zero for

[12] See Appendix B for further details about the data. Alternative specifications with money are estimated over different time periods due to constraints on data availability.

Figure 3: U.S. quarterly data



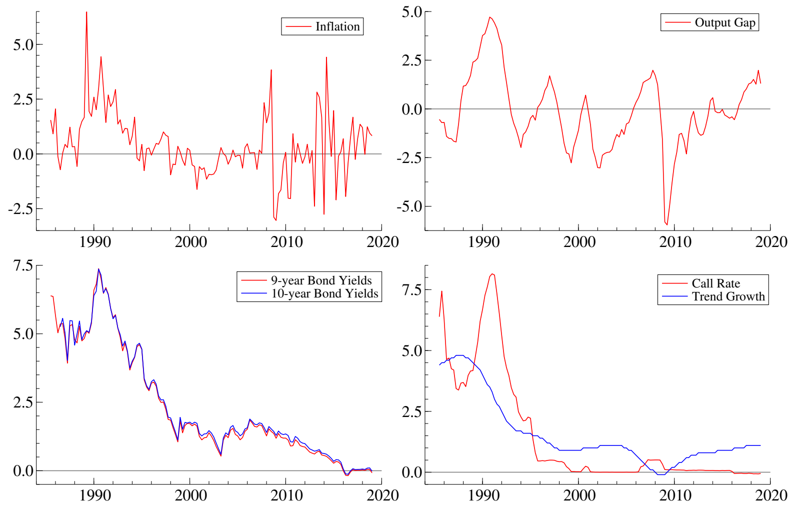
institutional reasons.

For Japan, we use quarterly data for core CPI inflation, a measure of the output gap provided by the Bank of Japan, and the Call Rate. In addition, we use two alternative measures for long yields: the 9-year and the 10-year government bond yields, which are available for different sample periods. The data sources are the Bank of Japan for the output gap and the Call Rate, the Ministry of Finance for the 9-year and the 10-year government bond yields, and Statistics Bureau of Japan for core CPI inflation. The available sample is from 1985q3 to 2019q1 if we include the 9-year government bond yields in the VAR, which is our baseline case, and from 1987q4 to 2019q1 if we use the 10-year yield. Following Hayashi and Koeda (2019), we set the ELB to track the interest on reserves (IOR).[13] For the sample period 1985q3-2019q1, the Call Rate is at the ELB for 49% of the observations. Following Hayashi and Koeda (2019), we use a trend growth series to account for the declining equilibrium real interest rate in Japan during the 1990s.[14] Figure 4 plots these series.

[13]Specifically, ELB = IOR + 7bp, which is slightly higher than Hayashi and Koeda (2019) who use IOR+5bp, in order to treat 2016q1 as being at the ELB.

[14]Specifically, we use the annual average growth rate of potential GDP as an additional control in our model. See Hayashi and Koeda (2019, pp. 1081–1083) for an extended discussion of this issue and its implications.

Figure 4: Japanese quarterly data



5 Empirical results

This section reports the main empirical results of the paper. We start by formalizing and testing the hypothesis that the ELB has been empirically irrelevant in each country, and then we estimate the (partially-identified) impulse responses to monetary policy shocks over time to gauge the effectiveness of UMP relative to conventional policy.

5.1 Tests of the ELB irrelevance hypothesis

Several studies assess the implications of the ELB for the effectiveness of monetary policy by comparing the responses of key variables across ELB and non-ELB regimes to a monetary policy shock. If the responses are sufficiently similar across the two regimes, the ELB is irrelevant for the effectiveness of monetary policy. In this subsection, we provide formal tests of the irrelevance hypothesis (IH), based on the methodology discussed in Section 2.

The first approach to test the IH is motivated by Swanson (2018) and Debortoli et al. (2019), who show that monetary policy remains similarly effective across ELB and non-ELB regimes, and establish that long-term interest rates are a plausible indicator of the stance of monetary policy. These authors develop SVARs that include long-term, rather than short-term interest rates as indicators of monetary policy. They use such VARs to identify the impulse responses of the macroeconomic variables to monetary policy as well as the response of policy to economic conditions, and find that those responses are similar across ELB and non-ELB regimes in the U.S. The implicit and testable assumption that underlies their analysis is that the short-term interest rate can be excluded from the dynamics of all the other variables in the system, and the dynamics of the system do not change as the economy enters an ELB regime. This was also formally established in part (ii) of Proposition 1 using the DSGE model developed in Section 3, though it is important to stress that the above testable implications of IH must hold under any theoretical model of UMP, so they are not predicated to the specific DSGE model of Section 3. This hypothesis can be tested as an exclusion restriction in an SVAR that includes both the short and the long rates. Since the short rate is subject to a binding ELB constraint, the relevant framework is the CKSVAR and the special case of KSVAR introduced in Section 2. Specifically, looking at the reduced-form specification of the model in equation (5), the IH can be formulated as:

$$\mathrm{IH}_1 : C_{12} = C_{12}^{*} = 0 \text{ and } \widetilde{\beta} = 0. \tag{14}$$

In words, $C_{12} = C_{12}^{*} = 0$ means that lags of the short-rate (Y_{2t}) and the shadow rate (Y_{2t}^{*}) can be excluded from the equations (5) that determine the remaining variables (Y_{1t}) in the VAR, and $\widetilde{\beta} = 0$ means that the slope coefficients and the variance of the errors of those equations (for Y_{1t}) remain the same when the economy moves across regimes.

The results of the likelihood ratio test of the null hypothesis IH_1 in (14) are reported in Table 1. Panel A reports results based on a KSVAR model, in which lags of the shadow rate Y_{2t}^{*} do not appear on the right hand side of equations (1) and (2). In this case, we test the null hypothesis (14) against an alternative hypothesis that imposes $C_{12}^{*} = 0$, so this test only has power against violation of the remaining restrictions in (14). We do so because the KSVAR model is simpler to estimate for the absence of latent variables, and a rejection of $C_{12} = 0$ would suffice to reject the null hypothesis (14). Panel B reports the results using the general CKSVAR. The table reports results for specifications with different lag lengths of the VAR(p), where $p = 1, \ldots, 5$. Column pv-p reports the p-value of a test for selecting the number of lags in the model,[15] which is an alternative approach to the Akaike Information Criterion (AIC), also reported in the Table. Both measures consistently select three lags for the U.S. and two lags for Japan. Column p-val reports the asymptotic p-value of our test of (14). It shows the data strongly reject the exclusion restrictions implied by the IH for both countries and in both KSVAR and CKSVAR specifications.

[15] It is the asymptotic p-value of a LR test of (C)KSVAR(p) against (C)KSVAR($p+1$).

Table 1: Test for excluding short rates from VAR that includes long rates

Panel A: KSVAR												
	United States						Japan					
p	loglik	pv-p	AIC	LR	df	p-val	loglik	pv-p	AIC	LR	df	p-val
5	-210.8	-	2.60	52.52	18	0.000	248.1	-	-2.18	27.82	18	0.065
4	-220.0	0.295	2.54	49.16	15	0.000	239.9	0.425	-2.30	28.10	15	0.021
3	-232.9	0.072	2.51	41.55	12	0.000	232.2	0.471	-2.42	28.58	12	0.004
2	-264.9	0.000	2.65	42.27	9	0.000	223.8	0.445	-2.53	25.71	9	0.002
1	-295.1	0.000	2.77	33.41	6	0.000	184.8	0.000	-2.19	32.32	6	0.000
Panel B: CKSVAR												
p	loglik	pv-p	AIC	LR	df	p-val	loglik	pv-p	AIC	LR	df	p-val
5	-188.2	-	2.58	84.12	33	0.000	284.7	-	-2.42	90.39	33	0.000
4	-200.9	0.185	2.51	73.39	27	0.000	277.1	0.766	-2.61	91.55	27	0.000
3	-218.5	0.019	2.49	57.07	21	0.000	258.1	0.081	-2.62	73.52	21	0.000
2	-254.6	0.000	2.63	49.97	15	0.000	242.1	0.018	-2.68	56.16	15	0.000
1	-287.9	0.000	2.74	37.98	9	0.000	204.8	0.000	-2.43	63.03	9	0.000

Note: Panel A reports results for a KSVAR(p) with inflation, output gap, long rate and policy rate. Panel B reports corresponding results for a CKSVAR(p) that includes shadow rates. Estimation sample is 1960q1-2019q1 for the U.S. and 1985q3-2019q1 for Japan. Long rates are 10-year government bond yields for the U.S. and 9-year yields for Japan. loglik is the value of the log-likelihood. pv-p is the p-value of the test for lag reduction. AIC is the Akaike information criterion. LR test statistic for excluding short rates from equations for inflation, output gap and long rates. df is number of restrictions. Asymptotic p-values reported.

Note that the hypothesis that lags of the short rate can be excluded from a VAR that includes long rates can also be tested in a standard VAR framework that includes lags of the short rate as external regressors. This would correspond to testing IH_1 against an alternative hypothesis that fixes $\tilde{\beta} = 0$ and $C_{12}^* = 0$, i.e., it has no power against any changes in the dynamics that do not come directly through the lags of the short rate. This would be a weaker test of IH.

The second test of the IH is motivated by Proposition 1, which shows that when UMP is fully effective in overcoming the ELB, the dynamics of the economy can be adequately represented by a linear SVAR in Y_{1t} and Y_{2t}^*. This hypothesis is represented by a VAR that entails pure censoring and no kink, which we denoted CSVAR in Section 2. The CSVAR is a special case of the CKSVAR in equations (5)-(6) that arises when we impose the following testable restrictions:

$$\text{IH}_2 : C_{12} = 0, C_{22} = 0 \text{ and } \widetilde{\beta} = 0. \tag{15}$$

We test the null hypothesis IH_2 in (15) again using a likelihood ratio test. As for our first

Table 2: Testing CSVAR against CKSVAR

Country	p	LR	df	p-val
U.S.	3	25.63	15	0.042
Japan	2	51.02	11	0.000

Note: The unrestricted model is a CKSVAR(p) in inflation, output gap, long rate and policy rate. Long rate: 10-year government bond yield (U.S.), 9-year government bond yield (Japan). Policy rate: Federal Funds Rate (U.S.), Call Rate (Japan). Sample: 1960q1-2019q1 (U.S.), 1985q3-2019q1 (Japan). p chosen by AIC. LR test statistics of the restrictions that the model reduces to CSVAR(p). df is number of restrictions, asymptotic p-values reported.

test of IH_1 in (14), this new test does not rely on any specific model of UMP, such as the one in Section 3, because any VAR that includes short rates must admit a CSVAR representation with constant parameters across regimes when the irrelevance hypothesis holds. Otherwise, the ELB would result in observable changes across regimes violating the hypothesis that the ELB is empirically irrelevant. As explained in Section 2, this model abstracts from any direct measures of UMP, and instead internalizes UMP through the shadow rate Y_{2t}^*. We perform the test with the three core observables, inflation and output gap in Y_{1t}, and the short-term policy rate in Y_{2t}, and we also include the long rate in Y_{1t} for robustness.

Table 2 reports the results of the likelihood ratio test of the null hypothesis IH_2 in (15) for the U.S. (top row) and Japan (bottom row). We include 3 lags for the U.S. and 2 lags for Japan, according to the AIC.[16] The results show that the IH is rejected for both economies at the 5% level of significance.

Robustness checks. We check the robustness of our results to possible omission of alternative channels of unconventional monetary policy, by including money growth in the Y_1 variables of the VAR that we use to test the null hypothesis IH_1 in (14). Using several different monetary aggregates for the U.S., we consistently reach the same conclusion: the IH is firmly rejected, see Table 5 in Appendix C.

We also check the robustness of the U.S. results to the well-documented fall in macroeconomic volatility in the mid-1980s, known as the Great Moderation, as well as a possible change in monetary policy regime occurring at that time by performing the above tests over the subsample 1984q1-2019q1. Our conclusions remain the same: the IH is firmly rejected, see Tables 6 and 7 in Appendix C.

[16]Table 1 reports the AIC for several alternative specifications of the model.

Table 3: Test for excluding long rates from VAR that includes short rates

Country	p	LR	df	p-val
U.S.	3	4.671	6	0.587
Japan	2	8.981	4	0.062

Note: Unrestricted model is CKSVAR(p) in inflation, output gap, policy rate, and long rate. p chosen by AIC. Restricted model excludes lags of long rate from other equations. Policy rate: Federal Funds Rate (U.S.) or Call Rate (Japan); long rate: 10-year bond yield (U.S.) or 9-year bond yield (Japan). Sample: 1960q1-2019q1 (U.S.) or 1985q3-2019q1 (Japan).

Finally, we find that the results continue to hold for Japan if we use the 10-year government bond yield that is available from 1987q3. The findings are reported in Tables 8 and 9 in Appendix C.

5.2 Tests of the (ir)relevance of long rates

The statistical tests in the previous section reject the IH of the ELB, and thus the possibility of excluding the short rate by controlling for the long rate. We now assess whether movements in the short-term interest rate, including its shadow value during ELB regimes, are sufficient to encapsulate the effect of both conventional and unconventional monetary policies, and therefore test the exclusion restriction on the long rate from the SVAR models. Proposition 2 in Section 3 shows that the exclusion restriction on the long-run rate holds in the theoretical model.

We perform this test in the CKSVAR model that includes inflation, output gap, the long rate, and the short (policy) rate.[17] The null hypothesis is that lags of the long rate can be excluded from the three equations for inflation, output gap, and the short rate. Table 3 reports the results. It shows that we cannot reject the null hypothesis that the long rate can be excluded from the model at the 5% level of significance in both countries. In the remainder of this section, we will therefore omit the long rate and use a three-equation model to study the impact of monetary policy.

[17]Results for the corresponding more restrictive KSVAR specification are reported in Table 10 in Appendix C.

5.3 Impact of monetary policy

Our results establish that the dynamics of the two economies are different across the ELB and non-ELB regimes, and lead us to conclude that the ELB is empirically relevant. Our findings imply that the responses of the economy to monetary policy are different across regimes, but they are silent on the magnitude of the differences. In this subsection, we address this issue using identified impulse responses from the CKSVAR models introduced earlier.

Since the model is nonlinear, the impulse responses functions (IRFs) are state-dependent. We will follow the approach in Koop et al. (1996), already used in Section 3.3, according to which the IRF to a monetary policy shock of magnitude ς is given by the difference in the expected path of the endogenous variables when the policy shock takes the value ς, versus the path when the shock is zero, conditional on the state of the economy prior to the shock. This approach is the most commonly used in the literature, see, e.g., Hayashi and Koeda (2019). In our model, there is an additional complication that lagged shadow rates are unobserved, so we evaluate the IRFs at the smoothed estimates of those latent variables, that is, our IRFs are given by:

$$IRF_{h,t}\left(\varsigma, X_t, \widehat{\overline{X}}_t^*\right) = E\left(Y_{t+h}|\varepsilon_{2t} = \varsigma, X_t, \widehat{\overline{X}}_t^*\right) - E\left(Y_{t+h}|\varepsilon_{2t} = 0, X_t, \widehat{\overline{X}}_t^*\right), \qquad (16)$$

where $\widehat{\overline{X}}_{t,j}^*$ is the smoothed estimate of the state vector $\overline{X}_{t,j}^*$ when it is unobserved.[18]

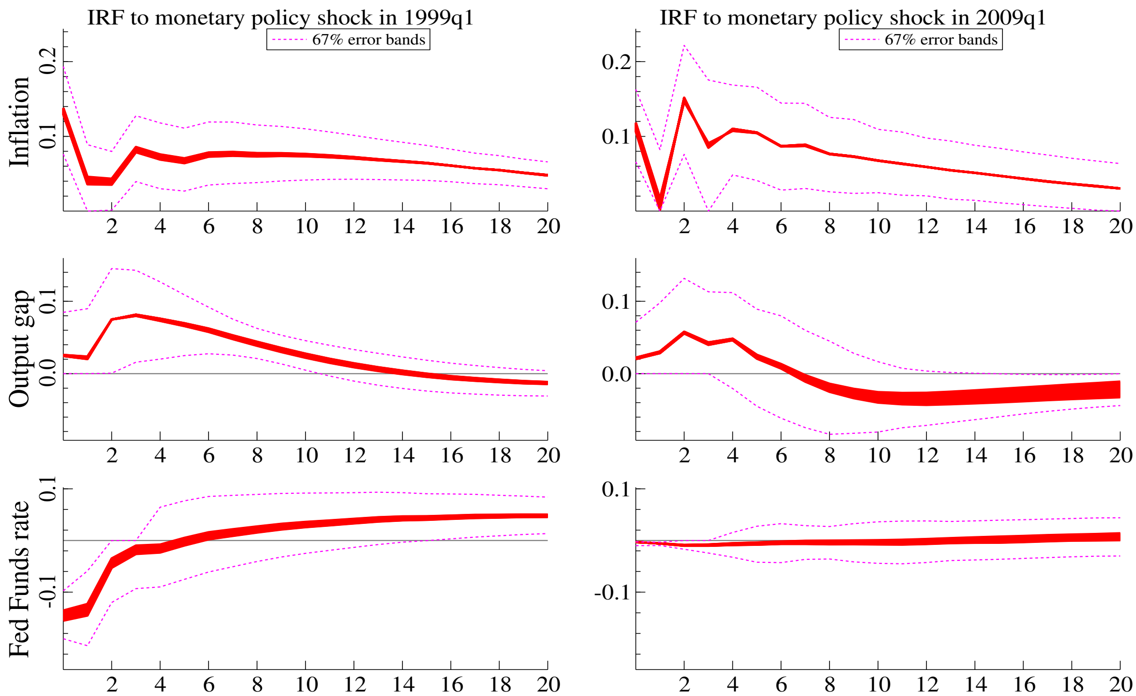
As explained in Section 2, the IRFs are generally set-identified unless we assume there is no contemporaneous effect of UMP on Y_1, which corresponds to setting $\xi = 0$ in the CKSVAR model. We will not be imposing such an assumption in our analysis. We proceed by first obtaining the identified set on ξ, β and γ by solving equations (7) and (8) at the estimated values of $\widetilde{\beta}$ and Ω, as explained in Section 2 (see the discussion following equations (7) and (8)), and then simulating the model paths at each of the values of the structural parameters in the identified set.[19]

The estimation results for the parameter ξ are as follows. Recall that the parameter ξ determines the impact effect of UMP, where the two limiting cases of $\xi = 0$ and $\xi = 1$ correspond to UMP being completely ineffective on impact and as effective as conventional policy in non-ELB regimes on impact, respectively. When we restrict the range of ξ to

[18] $\overline{X}_{t,j}^* = \min\left(Y_{2t-j}^* - b_t, 0\right)$ for $j = 1, ..., p$, where p is the order of the VAR.

[19] The algorithm for obtaining the identified set is provided in Mavroeidis (2021).

Figure 5: Impulse responses to a monetary policy shock in the U.S.



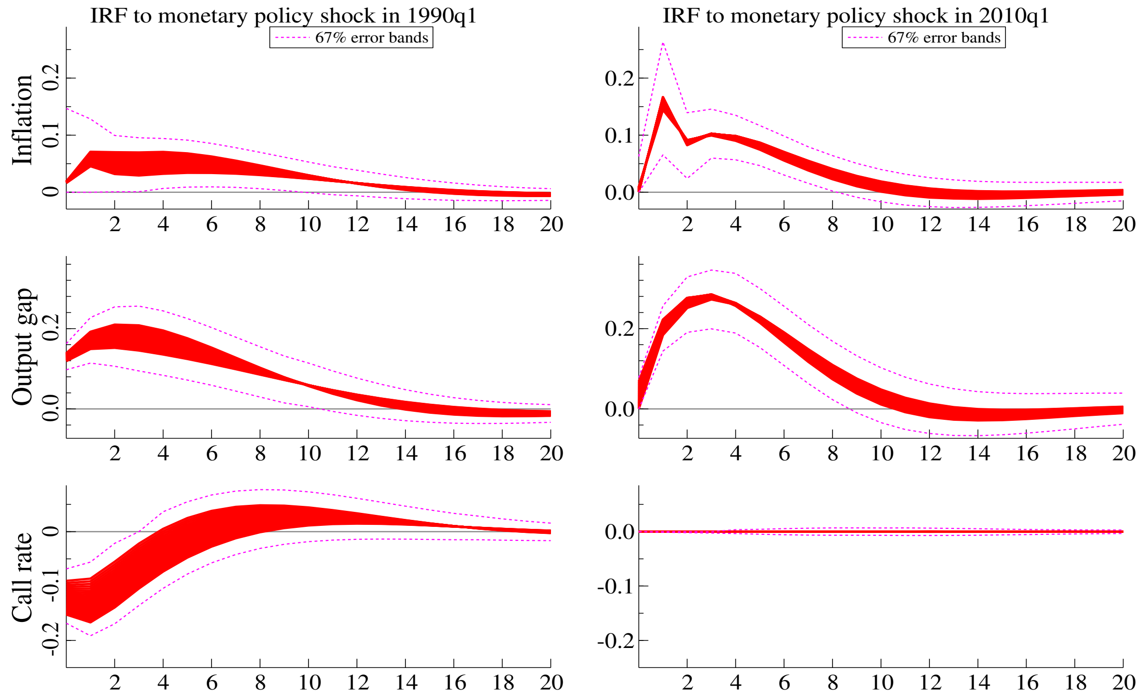
Note: Identified sets of IRFs in 1999q1 and 2009q1 to a -25bps monetary policy shock estimated from CKSVAR(3) model in inflation, output gap, and the Federal Funds Rate for the U.S. over the period 1960q1-2019q1, identified by the sign restrictions that the shock has nonnegative effects on inflation and output and nonpositive effects on the short rate up to four quarters. Dotted lines give 67% error bands.

$[0, 1]$ and impose no further identifying restrictions, the identified set for ξ is $[0, 0.78]$ for the U.S. and $[0, 0.34]$ for Japan. We sharpen the identified sets by using sign restrictions. We follow Debortoli et al. (2019) and impose the restrictions that a negative monetary policy shock should have a nonnegative effect on inflation and output, and a nonpositive effect on interest rates at a one-year horizon.[20] These sign restrictions clearly hold in the DSGE model developed in Section 3 (see Figure 2). With these sign restrictions, the identified set for the parameter ξ narrows down substantially for the U.S., from $[0, 0.78]$ to $[0.74, 0.76]$. For Japan the impact of the sign restrictions is more modest, from $[0, 0.34]$ to $[0, 0.26]$.

Figures 5 and 6 show the corresponding identified sets for the IRFs for the U.S. and Japan, respectively. The figures report identified IRFs of inflation, the output gap and the

[20] Note that because IRFs are state-dependent, these sign restrictions need to be imposed for all values of the initial states. In principle, this means working out the worst cases over the support of the distribution of the variables. However, a very similar conservative estimate of the identified set can be obtained if we simply impose the sign restrictions in every period.

Figure 6: Impulse responses to a monetary policy shock in Japan



Note: Identified sets of IRFs in 1990q1 and 2010q1 to a -25bps point monetary policy shock estimated using a CKSVAR(2) model in inflation, output gap and the Call Rate for Japan over the period 1985q3-2019q1, identified by the sign restrictions that the shock has nonnegative effects on inflation and output and nonpositive effects on the short rate up to 4 quarters. Dotted lines give 67% error bands.

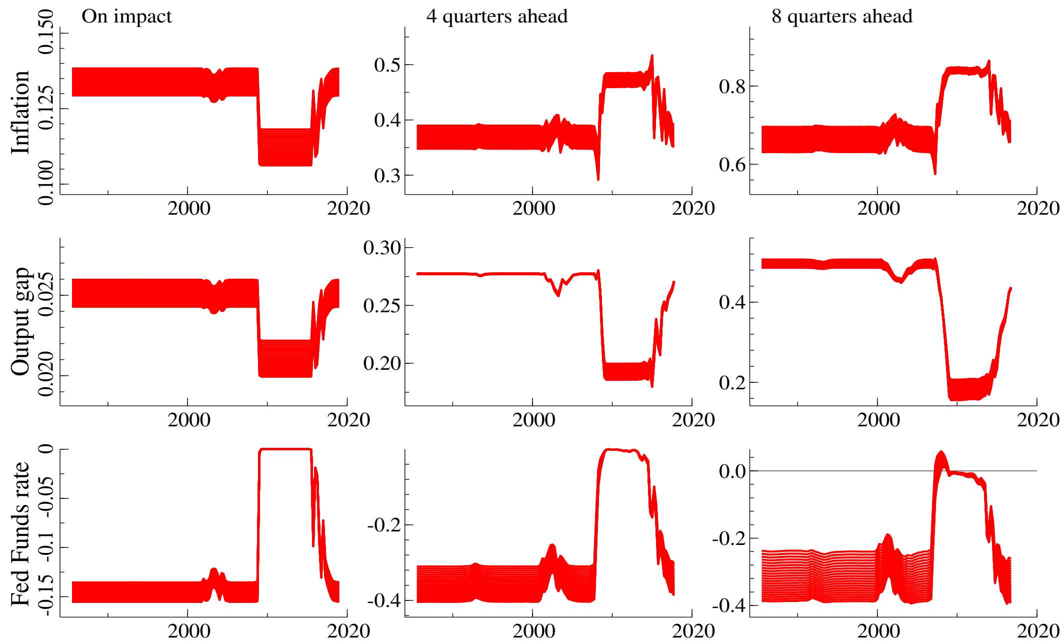
policy rate to a -25 basis points monetary policy shock.[21] The IRFs are computed at two different dates: the left panels report IRFs at dates when interest rates are well above the ELB (1999q1 for the U.S. and 1990q1 for Japan); the right panels report IRFs at dates when interest rates are at the ELB (2009q1 for the U.S. and 2010q1 for Japan). The policy effects differ across the two periods. For both countries, the (conventional) monetary policy shock has a bigger contemporaneous effect on all variables in the pre-ELB dates than the corresponding (unconventional) policy shock during the ELB dates, and the difference is larger in Japan than in the U.S.[22] However, in Japan the impulse responses to UMP appear to be stronger a few quarters out.[23]

[21]The figure also reports asymptotic confidence intervals obtained using the method of Imbens and Manski (2004), where we also impose the sign restrictions on the confidence bands, as in Granziera et al. (2018).

[22]This is because ξ is estimated lower in Japan than in the U.S.

[23]The reason why the delayed effects of UMP can be stronger than conventional policy even though $\xi < 1$ is because in the empirical model the coefficients on the lags of the shadow rate are completely unrestricted. This is more general than the theoretical model of Section 3 with the monetary policy rule (12), where the coefficient on the lagged shadow rate was restricted to be a constant fraction λ^* of the coefficient on

Figure 7: Responses to monetary policy shock in the U.S. over time



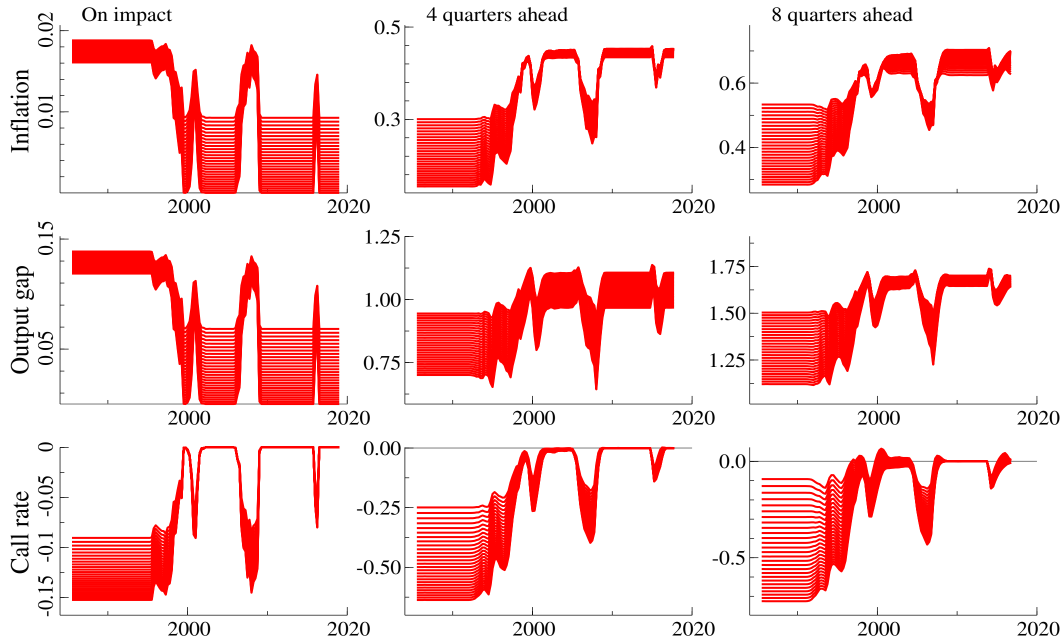
Note: Identified sets of (cumulative) impulse responses to a negative -25bps monetary policy shock at three different horizons, on impact, one year out and two years out, estimated using a CKSVAR(3) model in inflation, output gap, and the Federal Funds Rate for the U.S. over the period 1960q1-2019q1, identified by the sign restrictions that the shock has nonnegative effects on inflation and output and nonpositive effects on the short rate up to 4 quarters.

To shed further light on the differences in the responses over time, and in light of the fact that the IRFs are time-varying, we look at the evolution of the impulse responses at given horizons, 0, 4 and 8 quarters, over time. The results are reported in Figures 7 and 8 for the U.S. and Japan, respectively. In each figure, the panels on the left column report the impact effects of a -25 basis point monetary policy shock at each quarter from 1985q3 till the end of our sample. The panels in the middle column have the cumulative impulse responses after a year, while the panels on the right column give the corresponding cumulative responses after two years. We discuss each country in turn.

In the U.S. (Figure 7), we see a clear drop in the impact effect of policy during the ELB period relative to the pre-ELB period. The relative difference in the effectiveness of policy on both inflation and output on impact is over 20%. For the output gap, this difference remains,

the lagged policy rate above the ELB. The result of that restriction was that $\lambda^* < 1$ restricted UMP to have a uniformly weaker effect than conventional policy over all horizons. We did not wish to impose this overidentifying restriction in the empirical analysis.

Figure 8: Responses to monetary policy shock in Japan over time



Note: Identified sets of (cumulative) impulse responses to a -25bps monetary policy shock at three different horizons, on impact, one year out and two years out, estimated using a CKSVAR(2) model in inflation, output gap and the Call Rate for Japan over the period 1985q3-2019q1, identified by the sign restrictions that the shock has nonnegative effects on inflation and output and nonpositive effects on the short rate up to 4 quarters.

and increases one and two years ahead. However, the effect on inflation is the reverse: the cumulative effect of UMP on inflation is much stronger one and two years ahead. Therefore, UMP in the U.S. seems to have had a delayed but strong effect on inflation, but has been persistently less effective on output than conventional policy.

In Japan (Figure 8), there is a clear drop in the contemporaneous effect of policy on inflation and output during the ELB periods. There are three distinguishable ELB periods, 1999q2-2000q2, 2001q2-2006q2, and 2009q1 to the end of the sample. The contemporaneous policy effect on inflation is negligible, but the delayed effect one and two years later is stronger during the ELB periods than outside them. Like for the U.S., there seems to be a stronger delayed effect of UMP on inflation in Japan. Turning to the policy effect on output, we see that UMP is more than 50% weaker on impact, but catches up within one year, and stays stronger two years out. So, unlike the U.S., where UMP was less effective at all horizons, in Japan, this is not the case.

In sum, in the U.S. the response of inflation to UMP has been stronger than to conventional policy one or two years ahead, while it has been weaker for output over the same horizon. In Japan, UMP has had weaker effects than conventional policy on inflation and output on impact, but has had stronger delayed effects on these variables. In general, the findings show that the effects of UMP have been different across time, lending further support to our earlier finding that the ELB has been empirically relevant in both countries over that period.

Robustness to number of lags in the VAR. The estimation results for Japan are based on second-order VAR selected by the AIC and the sequential LR tests. Because these criteria are known to lead to overfitting and our sample for Japan is relatively small (34 years), we investigated the robustness of the above results to a first-order VAR, and found that our conclusions continue to hold.[24]

Relationship to Hayashi and Koeda (2019). Our analysis of the Japanese data draws heavily on the seminal contribution of Hayashi and Koeda (2019). Specifically, the use of trend growth that they proposed to control for the decline in the short rate over our sample is essential to get a VAR that satisfies the sign restrictions on the IRFs. There are several apparent methodological differences between our papers. Hayashi and Koeda (2019) use monthly data over a shorter period 1992-2012, while we use quarterly data from 1985 to 2019. They model QE via excess reserves and FG via an exit condition on inflation, while we rely on the shadow rate to capture both forms of UMP, motivated by the theoretical model of Section 3. They use recursive identification, assuming that inflation and output are predetermined, while we do not, and rely instead on sign restrictions and the changes in the dynamics and variances across regimes for identification.[25] However, these differences are not as large as they appear. For example, their results show that inflation and output are not predetermined at quarterly frequency, which is consistent with our findings, since the identified impact effect of monetary policy easing on inflation and output is positive within the quarter. Both models have regime-dependent decision rules that are fairly similar when translated to quarterly frequency.[26] And finally, even though they provide convincing

[24] Results available through our replication code.

[25] Hayashi and Koeda (2019) impose exclusion restrictions on the dynamics of the policy reaction function, while we do not.

[26] Our approach imposes continuity in the decision rules across regimes, while Hayashi and Koeda (2019) do not.

documentation that an inflation exit condition fits better the narrative of Japanese monetary policy over their sample period, the use of a Reifschneider and Williams (2000) shadow-rate FG rule appears nearly observationally equivalent to an inflation exit condition because of the relative scarcity of movements in and out of the ELB regime over the sample. This explains why our conclusions are broadly consistent with theirs.[27]

Finally, our theoretical model abstracts from possible negative effects of UMP such as those of "the reversal interest rate" (Brunnermeier and Koby (2019)), and our empirical analysis excludes those effects through sign restrictions. If we remove the sign restrictions during ELB regimes and maintain them during non-ELB regimes, we can allow ξ to be negative, which can capture policy reversals on impact. It turns out that removing the sign restrictions during the ELB periods does not affect the identified set for ξ for the impulse responses for the U.S., while the effect for Japan is limited.[28]

5.4 Shadow rates

We conclude this section by cautiously reporting our models' estimates of the shadow rates for each country. The important caveat that needs to be borne in mind in interpreting those figures is that the shadow rates are not identified under our present assumptions. As explained in Mavroeidis (2021), identifying the shadow rate Y_{2t}^* in the empirical model (1)-(3) requires knowledge of the parameter α, which scales the reaction function coefficients and policy shocks during the ELB regimes and is not identified without additional information. This parameter is needed *in addition* to the parameter ξ that measures the overall impact effect of UMP. In other words, to properly identify the shadow rate and interpret it as a measure of desired policy stance, we need to be able to isolate the effect of FG encapsulated by α. This exercise is beyond the scope of the present paper.

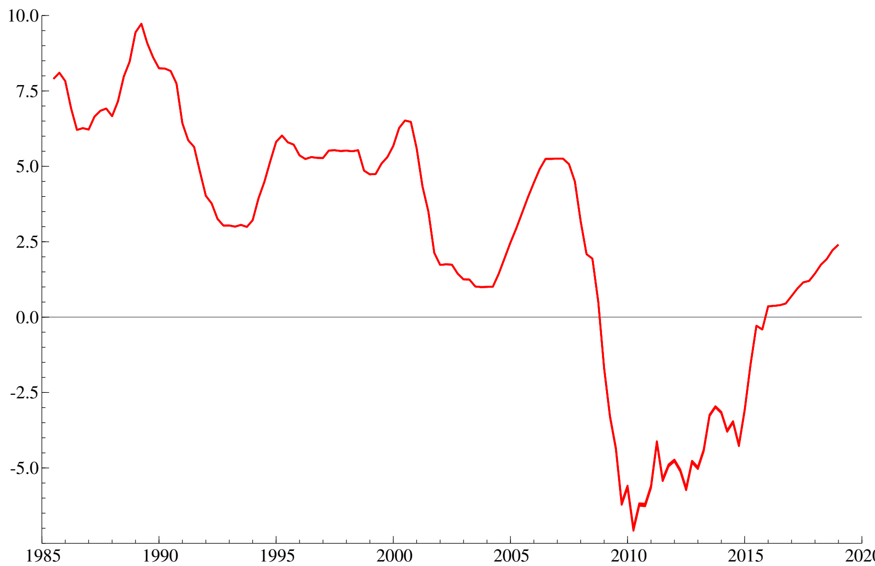
With the above caveat in mind, we report identified shadow rates under the assumption of $\alpha = 0$. The shadow rates are given in Figures 9 and 10 for the U.S. and Japan, respectively. Different values of α would scale those estimates by a factor $1 + \alpha$.[29] Note that, even with $\alpha = 0$, the shadow rate is only partially identified because it also depends on the parameter

[27]For example, the rejection of the irrelevance hypothesis (14) for Japan is due to both $\widetilde{\beta} \neq 0$ and $C_{12}^* \neq 0$. This accords with Hayashi and Koeda (2019), who report significant changes both in the constant term as well as the coefficient on the lag of the policy variable across regimes.

[28]The identified set for ξ for Japan becomes $[-0.08, 0.29]$, which includes negative responses of inflation and output to an expansionary UMP shock on impact. However, the delayed responses are not significantly affected, and thus any possible negative effects of UMP are short-lived.

[29]Results are available on request.

Figure 9: Shadow policy rate for the U.S.



Note: Estimated using a CKSVAR(3) model in inflation, output gap, and the Federal Funds Rate for the U.S. over the period 1960q1-2019q1 (plotted over the sub-sample 1985q3-2019q1), identified by the sign restrictions that a -25bp monetary policy shock has nonnegative effects on inflation and output and nonpositive effects on the short rate up to 4 quarters.

ξ that is partially identified. This uncertainty due to ξ is reflected in the shaded areas below the ELB in the figures.[30] In the case of the U.S., the shadow rate dropped sharply soon after the onset of the great financial crisis of 2008. It reached its smallest value at the beginning of 2010 and gradually recovered until the exit from the ELB in 2016. In Japan, the behaviour of the shadow rate is different during the three ELB episodes. During the first episode, the shadow rate fell modestly. In the second episode, it exhibited a persistent decline until the beginning of 2005, followed by a quick reversal. In the third episode, which coincided with the ELB in the U.S., the decline was sharp, and followed by a second wave of declines that lasted until mid 2012. From that point on, the shadow rate exhibited a steady rise, but stayed far from zero even at the end of the sample, and remained near its trough in the second episode.

[30]The shadow rate is equal to the observed policy rate above the ELB, see eq. (3). Below the ELB, it is given by the equation $Y_{2t}^{*} = \kappa \overline{Y}_{2t} + (1-\kappa)b_t$, where $\kappa = (1+\alpha)(1-\gamma\beta)/(1-\xi\gamma\beta)$ and $\overline{Y}_{2t}$ is a "reduced-form" shadow rate that can be filtered from the data using the likelihood, see Mavroeidis (2021).

Figure 10: Shadow policy rate for Japan

Note: Estimated using a CKSVAR(2) model in inflation, output gap and the Call Rate for Japan over the period 1985q3-2019q1, identified by the sign restrictions that a -25bp monetary policy shock has nonnegative effects on inflation and output and nonpositive effects on the Call Rate up to 4 quarters.

6 Conclusion

The paper develops theoretical and empirical models to study the effectiveness of unconventional monetary policy. The theoretical model allows the degree of effectiveness of unconventional policy to range from being as fully effective as conventional policy to being completely ineffective, and it provides theoretical underpinnings to the empirical model. Our empirical analysis is based on an agnostic structural VAR model that accounts for the effective lower bound on the policy rate and captures unconventional policy via a shadow rate. Our results provide strong evidence against the hypothesis that the ELB is empirically irrelevant, which implies that the ELB has been an important constraint on monetary policy in both the U.S. and Japan. However, our results also reveal strong delayed effects and country-specific differences in the effect of unconventional policy relative to conventional policy.

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Appendix

A Theoretical Model

Appendix A presents a simple New Keynesian model with the ELB and UMP. The model is a simplified version of Chen et al. (2012), extended to incorporate FG in the spirit of Reifschneider and Williams (2000). To keep the analysis focused on the salient features of the transmission mechanisms of UMP, the model abstracts from capital accumulation, consumption habit formation, and various shocks. The model combines two types of unconventional monetary policies: QE – a central bank's purchase of long-term government bonds, activated when the economy hits the ELB; and FG. There are three shocks: a demand (preference) shock, a supply (productivity) shock, and a monetary policy shock.

A.1 Model building blocks

A.1.1 Long-term bonds

There is a long-term government bond (consol bond). The long-term bond issued at time t yields μ^{j-1} dollars at time $t+j$ over time. Let $R_{L,t+1}$ denote the gross nominal rate from time t to $t+1$. The period-t price of the bond issued at time t, $P_{L,t}$, is defined as

$$\begin{aligned} P_{L,t} &= E_t\left(\frac{1}{R_{L,t+1}} + \frac{\mu}{R_{L,t+1}R_{L,t+2}} + \frac{\mu^2}{R_{L,t+1}R_{L,t+2}R_{L,t+3}} + ...\right) \\ &= E_t\left(\frac{1}{R_{L,t+1}} + \frac{\mu}{R_{L,t+1}}P_{L,t+1}\right). \end{aligned} \tag{A.1}$$

The gross yield to maturity (or the long-term interest rate) at time t, $\bar{R}_{L,t}$ is defined as

$$E_t\left(\frac{1}{\bar{R}_{L,t}} + \frac{\mu}{\left(\bar{R}_{L,t}\right)^2} + \frac{\mu^2}{\left(\bar{R}_{L,t}\right)^3} + ...\right) = P_{L,t},$$

or

$$P_{L,t} = \frac{1}{\bar{R}_{L,t} - \mu}. \tag{A.2}$$

Let $B_{L,t|t-s}$ denote period-t bond holdings issued at time $t-s$. Suppose that a household owns $B_{L,t|t-s}$ for $s = 1, 2, ...$ in the beginning of period t. The total amount of dividends the household receives in period t is

$$\sum_{s=1}^{\infty}\mu^{s-1}B_{L,t|t-s}.$$

Note that having one unit of $B_{L,t|t-s}$ is equivalent to having μ^{s-1} units of $B_{L,t|t-1}$ because they both yield μ^{s-1} dollars. The total amount of dividends then can be expressed in terms of $B_{L,t|t-1}$ as

$$\sum_{s=1}^{\infty}\mu^{s-1}B_{L,t|t-s} \equiv B_{L,t-1},$$

where $B_{L,t-1}$ denotes the amount of bonds in units of the bonds issued at time $t-1$, held by the household in the beginning of period t. Let $P_{L,t|t-s}$ denote the time-t price of the bond issued at time $t-s$. Then, the value of all the bonds at time t is

$$\sum_{s=1}^{\infty}P_{L,t|t-s}B_{L,t|t-s}$$

Each price satisfies

$$\begin{aligned} P_{L,t|t-s} &= E_t\left(\frac{\mu^s}{R_{L,t+1}} + \frac{\mu^{s+1}}{R_{L,t+1}R_{L,t+2}} + \frac{\mu^{s+2}}{R_{L,t+1}R_{L,t+2}R_{L,t+3}} + ...\right) \\ &= \mu^s P_{L,t} \end{aligned}$$

Then the value of all the bonds at time t is

$$\sum_{s=1}^{\infty} P_{L,t|t-s} B_{L,t|t-s} = P_{L,t}\mu \sum_{s=1}^{\infty} \mu^{s-1} B_{L,t|t-s} = \mu P_{L,t} B_{L,t-1}.$$

So the return of holding $B_{L,t-1}$ is given by the sum of dividends and the value of all the bonds as:

$$B_{L,t-1} + \mu P_{L,t} B_{L,t-1} = (1 + \mu P_{L,t}) B_{L,t-1} = P_{L,t} \bar{R}_{L,t} B_{L,t-1} = \frac{\bar{R}_{L,t}}{\bar{R}_{L,t} - \mu} B_{L,t-1}.$$

A.1.2 Households

There are two types of households: unrestricted households (U-households) and restricted households (R-households). U-households, with population ω_u, can trade both short-term and long-term government bonds subject to a transaction cost ζ_t per unit of long-term bonds purchased. R-households, with population $\omega_r = 1 - \omega_u$, can trade only long-term government bonds. For $j = u, r$, each household chooses consumption c_t^j, hours worked h_t^j, government bond holdings $B_{L,t}^j$ and B_t^j to maximize utility,

$$\sum_{t=0}^{\infty} \beta_j^t d_t \left[\frac{\left(c_t^j\right)^{1-\sigma}}{1-\sigma} - \psi \frac{\left(h_t^j\right)^{1+1/\nu}}{1+1/\nu} \right],$$

subject to: for a U-household,

$$P_t c_t^u + B_t^u + (1+\zeta_t) P_{L,t} B_{L,t}^u = (1+i_{t-1}) B_{t-1}^u + P_{L,t} \bar{R}_{L,t} B_{L,t-1}^u + W_t h_t^u - T_t^u + \Pi_t^u,$$

and for a R-household,

$$P_t c_t^r + P_{L,t} B_{L,t}^r = P_{L,t} \bar{R}_{L,t} B_{L,t-1}^r + W_t h_t^r - T_t^r + \Pi_t^r,$$

where P_t is the price level and i_t is the short-term interest rate. In addition $\bar{R}_{L,t}$ denotes the gross yield to maturity at time t on the long-term bond

$$\bar{R}_{L,t} = \frac{1}{P_{L,t}} + \mu, \quad 0 < \mu \leq 1.$$

The average duration of the bond is given by $\bar{R}_{L,t} / \left(\bar{R}_{L,t} - \mu\right)$. There is a shock d_t to the preference, and it is given by:

$$d_t = \begin{cases} e^{z_1^b} e^{z_2^b} ... e^{z_t^b} & \text{for } t \geq 1 \\ 1 & \text{for } t = 0 \end{cases},$$

where z_t^b is a preference (demand) shock, which is assumed to follow the AR(1) process

$$z_t^b = \rho_b z_{t-1}^b + \epsilon_t^b,$$

with $\epsilon_t^b \sim$ i.i.d. $N\left(0, \sigma_b^2\right)$.

We assume that the transaction cost of trading long-term bonds for the U-households is collected by financial firms and redistributed as a lump-sum profits to the U-households. Under the assumption, the transaction cost does not appear in the good market clearing condition, which is given by:

$$y_t = \omega_u c_t^u + (1 - \omega_u) c_t^r. \tag{A.3}$$

Arranging the first-order conditions of the U-household's problem yields the following optimality conditions:

$$w_t = \psi \left(c_t^u\right)^\sigma \left(h_t^u\right)^{1/\nu}, \tag{A.4}$$

$$1 = E_t \beta_u e^{z_{t+1}^b} \left(\frac{c_{t+1}^u}{c_t^u} \right)^{-\sigma} \frac{1+i_t}{\pi_{t+1}}, \tag{A.5}$$

$$1 + \zeta_t = E_t \beta_u e^{z_{t+1}^b} \left(\frac{c_{t+1}^u}{c_t^u} \right)^{-\sigma} \frac{R_{L,t+1}}{\pi_{t+1}}, \tag{A.6}$$

where $w_t \equiv W_t/P_t$ denotes the real wage, $\pi_t \equiv P_t/P_{t-1}$ denotes the inflation rate, and $R_{L,t+1}$ denotes the annual yield of the long-term bond between periods t and $t+1$, given by

$$R_{L,t+1} \equiv \frac{P_{L,t+1}}{P_{L,t}} \bar{R}_{L,t+1} = \frac{P_{L,t+1}}{P_{L,t}} \left(\frac{1}{P_{L,t+1}} + \mu \right) = \frac{1 + \mu P_{L,t+1}}{P_{L,t}}.$$

Similarly, arranging the first-order conditions of the R-household's problem yields

$$w_t = \psi \left(c_t^r\right)^{\sigma} \left(h_t^r\right)^{1/\nu}, \tag{A.7}$$

$$1 = E_t \beta_r e^{z_{t+1}^b} \left(\frac{c_{t+1}^r}{c_t^r} \right)^{-\sigma} \frac{R_{L,t+1}}{\pi_{t+1}}, \tag{A.8}$$

A.1.3 Firms

The firm sector consists of two types of firms: final good firms and intermediate goods firms. The problems of these firms are standard except that the average discount rate between U-households and R-households is used in discounting the profits of these firms. The profits need to be derived explicitly because one of the two households' budget constraints constitutes an equilibrium condition as well as a good market clearing condition.

Competitive final good firms combine intermediate goods $\{y_t(l)\}_{l=0}^{1}$ and produce the final good y_t according to

$$y_t = \left[\int_0^1 y_t(l)^{\frac{1}{\lambda_p}} dl \right]^{\lambda_p}, \ \lambda_p > 1.$$

The demand function for the l-th intermediate good is given by

$$y_t(l) = \left(\frac{P_t(l)}{P_t} \right)^{\frac{\lambda_p}{1-\lambda_p}} y_t.$$

Intermediate goods firms use labor and produce intermediate goods according to

$$y_t(l) = e^{z_t^a} h_t(l)^{\theta}, \ 0 < \theta \leq 1.$$

where z_t^a is a productivity shock, which is assumed to follow:

$$z_t^a = \rho_a z_{t-1}^a + \epsilon_t^a,$$

with $\epsilon_t^a \sim$ i.i.d. $N\left(0, \sigma_a^2\right)$. Because there is no price dispersion in steady state, the aggregate output can be written up to the first-order approximation as:

$$\hat{y}_t = z_t^a + \theta \hat{h}_t, \tag{A.9}$$

where $\hat{y}_t$ and $\hat{h}_t$ denote the aggregate output and hours worked in terms of deviation from steady state. The total cost of producing $y_t(l)$ is equal to

$$W_t h_t(l) = W_t \left(\frac{y_t(l)}{e^{z_t^a}} \right)^{\frac{1}{\theta}}.$$

In each period, intermediate goods firms can change their price with probability ξ identically and independently across firms and over time. For each l, the l-th intermediate good firm chooses the price, $\tilde{P}_t(l)$, to maximize the discounted sum of profits,

$$\max_{\tilde{P}_t(l)} E_t \sum_{s=0}^{\infty} (\xi\delta)^s \bar{\Lambda}_{t+s|t} \left[P_{t+s}(l) y_{t+s}(l) - W_{t+s} \left(\frac{y_{t+s}(l)}{e^{z_{t+s}^a}} \right)^{\frac{1}{\theta}} \right],$$

subject to the demand curve,

$$y_{t+s}(l) = \left(\frac{P_{t+s}(l)}{P_{t+s}}\right)^{\frac{\lambda_p}{1-\lambda_p}} y_{t+s},$$

where

$$\begin{aligned}
&\delta = \omega_u \beta_u + (1-\omega_u)\beta_r, \\
&\bar{\Lambda}_{t+s|t} \equiv d_{t+s|t}\left(\omega_u \Lambda^u_{t+s|t} + (1-\omega_u)\,\Lambda^r_{t+s|t}\right), \\
&\Lambda^j_{t+s|t} = \left(\frac{c^j_{t+s}}{c^j_t}\right)^{-\sigma} \frac{1}{P_{t+s}}, \quad d_{t+s|t} = \begin{cases} 1 & \text{if } s=0 \\ e^{z^b_{t+1}} e^{z^b_{t+2}} ... e^{z^b_{t+s}} & \text{if } s = 1,2,... \end{cases} \\
&P_{t+s}(l) = \tilde{P}_t(l)\Pi^p_{t,t+s}, \\
&\Pi^p_{t+s|t} = \begin{cases} 1 & \text{if } s=0 \\ \prod_{k=1}^{s} (\pi_{t+k-1})^{\iota_p} (\pi)^{1-\iota_p} & \text{if } s=1,2,... \end{cases}.
\end{aligned}$$

Substituting the demand curve into the objective function yields

$$\max_{\tilde{P}_t(l)} E_t \sum_{s=0}^{\infty} (\xi\delta)^s \bar{\Lambda}_{t+s|t} \left[\tilde{P}_t(l)\Pi^p_{t+s|t} \left(\frac{\tilde{P}_t(l)\Pi^p_{t+s|t}}{P_{t+s}}\right)^{\frac{\lambda_p}{1-\lambda_p}} Y_{t+s} - W_{t+s}\left(\frac{\tilde{P}_t(l)\Pi^p_{t+s|t}}{P_{t+s}}\right)^{\frac{\lambda_p}{(1-\lambda_p)\theta}} \left(\frac{y_{t+s}}{e^{z^a_{t+s}}}\right)^{\frac{1}{\theta}} \right].$$

The first-order condition is

$$0 = E_t \sum_{s=0}^{\infty} (\xi\delta)^s \bar{\Lambda}_{t+s|t} \left[\frac{1}{1-\lambda_p} \Pi^p_{t+s|t} y_{t+s}(l) - W_{t+s} \frac{\lambda_p}{(1-\lambda_p)\theta} \left(\frac{y_{t+s}(l)}{e^{z^a_{t+s}}}\right)^{\frac{1}{\theta}} \frac{1}{\tilde{P}_t(l)} \right].$$

Since $\tilde{P}_t(l)$ does not depend on l, index l is omitted hereafter. Define $\tilde{p}_t \equiv \tilde{P}_t / P_t$ and

$$\tilde{\Pi}^p_{t+s|t} = \begin{cases} 1 & \text{if } s=0 \\ \prod_{k=1}^{s} \frac{(\pi_{t+k-1})^{\iota_p}(\pi)^{1-\iota_p}}{\pi_{t+k}} & \text{if } s=1,2,... \end{cases}$$

The first-order condition can be transformed as

$$\begin{aligned}
0 = E_t \sum_{s=0}^{\infty} (\xi\delta)^s \bar{\Lambda}_{t+s} P_{t+s} \Bigg[& \frac{1}{1-\lambda_p} \frac{\Pi^p_{t+s|t}}{P_{t+s}} \left(\tilde{p}_t \tilde{\Pi}^p_{t+s|t}\right)^{\frac{\lambda_p}{1-\lambda_p}} y_{t+s} \\
& - \frac{W_{t+s}}{P_{t+s}} \frac{\lambda_p}{(1-\lambda_p)\theta} \left(\tilde{p}_t \tilde{\Pi}^p_{t+s|t}\right)^{\frac{\lambda_p}{(1-\lambda_p)\theta}} \left(\frac{Y_{t+s}}{e^{z^a_{t+s}}}\right)^{\frac{1}{\theta}} \frac{1}{\tilde{P}_t} \Bigg],
\end{aligned}$$

The above equation can be written as:

$$\tilde{p}_t = \left(\frac{\lambda_p}{\theta} \frac{\omega_u K^u_{p,t} + (1-\omega_u) K^r_{p,t}}{\omega_u F^u_{p,t} + (1-\omega_u) F^r_{p,t}}\right)^{\frac{(1-\lambda_p)\theta}{\theta-\lambda_p}} \tag{A.10}$$

where for $j = r$ and u

$$F^j_{p,t} = (c^j_t)^{-\sigma} y_t + \xi\delta E_t e^{z^b_{t+1}} (\tilde{\Pi}^p_{t+1|t})^{\frac{1}{1-\lambda_p}} F^j_{p,t+1}, \tag{A.11}$$

$$K^j_{p,t} = (c^j_t)^{-\sigma} \left(\frac{y_t}{e^{z^a_t}}\right)^{\frac{1}{\theta}} w_t + \xi\delta E_t e^{z^b_{t+1}} (\tilde{\Pi}^p_{t+1|t})^{\frac{\lambda_p}{(1-\lambda_p)\theta}} K^j_{p,t+1}. \tag{A.12}$$

The aggregate price level evolves following

$$P_t = \left[\xi[(\pi_{t-1})^{\iota_p} (\pi)^{1-\iota_p} P_{t-1}]^{\frac{1}{1-\lambda_p}} + (1-\xi)\tilde{P}_t^{\frac{1}{1-\lambda_p}}\right]^{1-\lambda_p},$$

which can be written as

$$\tilde{p}_t = \left[\frac{1 - \xi (\tilde{\Pi}^p_{t|t-1})^{\frac{1}{1-\lambda_p}}}{1 - \xi} \right]^{1-\lambda_p}. \tag{A.13}$$

The conditions, (A.10)-(A.13), summarize the price setting behavior of intermediate goods firms.

The aggregate nominal profits earned by intermediate goods firms are given by:

$$\Pi^m_t = \int_0^1 \left(P_t(l)\, y_t(l) - W_t \left(\frac{y_t(l)}{e^{z^a_t}} \right)^{\frac{1}{\theta}} \right) dl = P_t y_t - W_t \left(\frac{y_t}{e^{z^a_t}} \right)^{\frac{1}{\theta}},$$

where the last equality holds up to the first-order approximation. Then, the aggregate real profits are given by $\pi^m_t = y_t - w_t \left(y_t / e^{z^a_t} \right)^{1/\theta}$.

A.1.4 Government

The government flow budget constraint is

$$(1 + i_{t-1})\, B_{t-1} + (1 + \mu P_{L,t})\, B_{L,t-1} = B_t + P_{L,t} B_{L,t} + T_t,$$

where $T_t = \omega_u T^u_t + (1 - \omega_u)\, T^r_t$. We assume that the lump-sum tax is imposed on households equally so that $T^u_t = T^r_t = T_t$. Without loss of generality, we assume that the amount of short-term bonds issued is constant at $b_t \equiv B_t / P_t = \bar{b}$. Then, the government flow budget constraint is reduced to:

$$(1 + \mu P_{L,t})\, B_{L,t-1} = P_{L,t} B_{L,t} + T_t.$$

A.1.5 Central bank

The nominal interest rate i_t set by the central bank is bounded below by the ELB of zero,

$$i_t = \max \{ i^*_t, 0 \} \tag{A.14}$$

where i^*_t is a shadow rate – the short-term rate the central bank would set if there were no ELB. The shadow rate i^*_t is given by[31]

$$i^*_t = i^{\text{Taylor}}_t - \alpha \left(i_t - i^{\text{Taylor}}_t \right). \tag{A.15}$$

The shadow rate i^*_t consists of two parts: i^{Taylor}_t and $\alpha(i_t - i^{\text{Taylor}}_t)$. First, i^{Taylor}_t is the Taylor-rule-based rate that responds to inflation π_t, output y_t, and the lagged "effective" interest rate $(1 - \lambda^*) i_{t-1} + \lambda^* i^*_{t-1}$:

$$i^{\text{Taylor}}_t - i = \rho_i \left((1 - \lambda^*) i_{t-1} + \lambda^* i^*_{t-1} - i \right) + (1 - \rho_i) \left[r_\pi \log(\pi_t / \pi) + r_y \log(y_t / y) \right] + \epsilon^i_t, \tag{A.16}$$

where ϵ^i_t is a monetary policy shock and variables without subscripts denote those in steady state. The parameter λ^* will be derived later in this appendix. Second, $\alpha(i_t - i^{\text{Taylor}}_t)$ in equation (11) encapsulates the strength of FG. A positive value for α will maintain the target rate i^*_t below the Taylor rate i^{Taylor}_t. Under the ELB of $i_t = 0$, the more the central bank has missed to set the interest rate at its Taylor rate, the lower the central bank sets its target rate i^*_t through equation (11) as long as $\rho_i \lambda^* > 0$ in equation (A.16).[32]

The central bank activates QE – long-term government bond purchases – once the economy hits the ELB. The central bank continues using the shadow rate as policy guidance under the ELB as in the case of positive interest rates. Specifically, the amount of long-term bond purchases depends on the shadow rate,

[31] Reifschneider and Williams (2000) employs the following rule: $i^*_t = i^{\text{Taylor}}_t - \alpha Z_t$ and $Z_t = \rho_Z Z_{t-1} + (i_t - i^{\text{Taylor}}_t)$ with $\rho_Z = 1$.

[32] Debortoli et al. (2019) consider the case of $\alpha = 0$ and $\lambda^* = 1$ in equation (A.16) and interpret ρ_i – the interest rate smoothing coefficient – as FG when i^*_t is below the ELB.

and as a result the amount of long-term government bonds, $b_{L,t} \equiv B_{L,t}/P_t$, held by the private agents is given by:

$$\widehat{b}_{L,t} = \begin{cases} 0 & \text{if } i_t^* > 0 \\ \gamma \frac{i_t^*}{1+i} & \text{if } i_t^* \leq 0 \end{cases}, \tag{A.17}$$

where a variable with hat denotes a deviation from steady state. This QE rule implies that asset purchases by the central bank is zero (relative to the steady state) when the ELB is not binding (i.e. $i_t = i_t^* > 0$) and, given $\gamma > 0$, such purchases are positive (i.e. $\hat{b}_{L,t} < 0$) when the shadow rate goes below zero (i.e. $i_t^* < 0$).

A.1.6 Market clearing and equilibrium

As well as the goods market clearing condition (A.3), there are market clearing conditions for labor, long-term government bonds, and short-term government bonds:

$$\omega_u h_t^u + (1-\omega_u)\, h_t^r = h_t, \tag{A.18}$$

$$\omega_u b_{L,t}^u + (1-\omega_u)\, b_{L,t}^r = b_{L,t}, \tag{A.19}$$

$$\omega_u b_t^u = b_t \tag{A.20}$$

Also, either the U-household's budget constraint or the R-household's budget constraint should be added as an equilibrium condition. Here the latter budget constraint is added:

$$c_t^r + P_{L,t} b_{L,t}^r = \left(\bar{R}_{L,t}/\pi_t\right) P_{L,t} b_{L,t-1}^r + w_t h_t^r - T_t^r/P_t + \Pi_t^r/P_t, \tag{A.21}$$

where

$$\frac{T_t^r}{P_t} = -\left(b_t + P_{L,t} b_{L,t}\right) + \frac{1+i_{t-1}}{\pi_t} b_{t-1} + \frac{1+\mu P_{L,t}}{\pi_t} b_{L,t-1},$$

$$\frac{\Pi_t^r}{P_t} = y_t - w_t h_t.$$

The cost of trading long-term bonds, ζ_t, is specified as

$$\zeta_t = \zeta \left(\frac{b_{L,t}}{b_L}\right)^{\rho_\zeta},$$

where $\rho_\zeta > 0$ and $\zeta > 0$ is the steady state value of ζ_t. The cost is increasing in the amount of long-term bonds relative to its steady state value, $\zeta_t' > 0$.

The system of equations for the economy consists of 19 equations, (A.3)-(A.21), with the following endogenous variables:

$$c_t^u, c_t^r, h_t^u, h_t^r, h_t, b_{L,t}^u, b_{L,t}^r, b_{L,t}, b_t^u, y_t, w_t, i_t, i_t^*, i_t^T, R_{L,t}, \pi_t, \tilde{p}_t, F_{p,t}^j, K_{p,t}^j.$$

A.2 Log-linearized equations

We log-linearize the equilibrium conditions of the theoretical model presented in Appendix A.1 around the steady state in which inflation is equal to the target rate of inflation set by the central bank. By doing so, we derive key equations in the system of equations (9)-(13) presented in Section 3.

Euler equation Log-linearizing equations (A.5), (A.6), (A.8) and (A.3), we obtain

$$0 = E_t\left[-\sigma\left(\hat{c}_{t+1}^u - \hat{c}_t^u\right) + \hat{\imath}_t - \hat{\pi}_{t+1} + z_{t+1}^b\right], \tag{A.22}$$

$$\frac{\zeta}{1+\zeta}\hat{\zeta}_t = E_t\left[-\sigma\left(\hat{c}_{t+1}^u - \hat{c}_t^u\right) + \hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b\right], \tag{A.23}$$

$$0 = E_t\left[-\sigma\left(\hat{c}_{t+1}^r - \hat{c}_t^r\right) + \hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b\right], \tag{A.24}$$

$$\hat{y}_t = \frac{\omega_u c^u}{y}\hat{c}_t^u + \frac{(1-\omega_u)\, c^r}{y}\hat{c}_t^r. \tag{A.25}$$

Equation (A.25) can be written as:

$$\hat{c}_t^u = \frac{y}{\omega_u c^u}\left\{\hat{y}_t - \frac{(1-\omega_u)\,c^r}{y}\hat{c}_t^r\right\}.$$

Subtracting $\hat{c}_{t+1}^u$ from $\hat{c}_t^u$ yields:

$$\begin{aligned}
\hat{c}_{t+1}^u - \hat{c}_t^u &= \frac{y}{\omega_u c^u}\left\{\hat{y}_{t+1} - \hat{y}_t - \frac{(1-\omega_u)\,c^r}{y}\left(\hat{c}_{t+1}^r - \hat{c}_t^r\right)\right\}, \\
&= \frac{y}{\omega_u c^u}\left\{\hat{y}_{t+1} - \hat{y}_t - \frac{(1-\omega_u)\,c^r}{y}\frac{\left(\hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b\right)}{\sigma}\right\}, \qquad \text{(A.26)}
\end{aligned}$$

where equation (A.24) was used in the second equality. Substituting equation (A.26) into equation (A.22) yields:

$$\begin{aligned}
0 &= E_t\left[-\sigma\left(\hat{c}_{t+1}^u - \hat{c}_t^u\right) + \hat{\imath}_t - \hat{\pi}_{t+1} + z_{t+1}^b\right], \\
&= E_t\left[-\frac{\sigma y}{\omega_u c^u}\left(\hat{y}_{t+1} - \hat{y}_t\right) + \frac{\sigma y}{\omega_u c^u}\frac{(1-\omega_u)\,c^r}{y}\frac{\left(\hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b\right)}{\sigma}\right. \\
&\left. + \hat{\imath}_t - \hat{\pi}_{t+1} + z_{t+1}^b\right],
\end{aligned}$$

or, by using equation (A.3) in steady state,

$$0 = E_t\left[-\sigma\left(\hat{y}_{t+1} - \hat{y}_t\right) + \frac{(1-\omega_u)\,c^r}{y}\hat{R}_{L,t+1} + \frac{\omega_u c^u}{y}\hat{\imath}_t - \hat{\pi}_{t+1} + z_{t+1}^b\right]. \qquad \text{(A.27)}$$

Also, substituting equation (A.26) into equation (A.23) yields:

$$\begin{aligned}
\frac{\zeta}{1+\zeta}\hat{\zeta}_t &= E_t\left[-\sigma\left(\hat{c}_{t+1}^u - \hat{c}_t^u\right) + \hat{R}_{L,t+1} - \hat{\pi}_{t+1}\right] \\
&= E_t\left[-\sigma\frac{y}{\omega_u c^u}\left\{\hat{y}_{t+1} - \hat{y}_t - \frac{(1-\omega_u)\,c^r}{y}\frac{\left(\hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b\right)}{\sigma}\right\}\right. \\
&\left. + \hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b\right], \\
&= E_t\left[-\frac{\sigma y}{\omega_u c^u}\left(\hat{y}_{t+1} - \hat{y}_t\right) + \frac{y}{\omega_u c^u}\left(\hat{R}_{L,t+1} - \hat{\pi}_{t+1} + z_{t+1}^b\right)\right],
\end{aligned}$$

or

$$\begin{aligned}
E_t\left(\hat{R}_{L,t+1} - \hat{\pi}_{t+1}\right) &= \sigma E_t\left(\hat{y}_{t+1} - \hat{y}_t\right) - E_t\left(z_{t+1}^b\right) + \frac{\omega_u c^u}{y}\frac{\zeta}{1+\zeta}\hat{\zeta}_t \\
&= \sigma E_t\left(\hat{y}_{t+1} - \hat{y}_t\right) - E_t\left(z_{t+1}^b\right) + \frac{\omega_u c^u}{y}\frac{\zeta}{1+\zeta}\rho_\zeta \widehat{b}_{L,t}. \qquad \text{(A.28)}
\end{aligned}$$

Combining equations (A.27) and (A.28) yields:

$$\begin{aligned}
0 &= E_t\left[-\sigma\left(\hat{y}_{t+1} - \hat{y}_t\right) + \frac{(1-\omega_u)\,c^r}{y}\hat{R}_{L,t+1} + \frac{\omega_u c^u}{y}\hat{\imath}_t - \hat{\pi}_{t+1} + z_{t+1}^b\right] \\
&= E_t\left[-\sigma\left(\hat{y}_{t+1} - \hat{y}_t\right) + \frac{(1-\omega_u)\,c^r}{y}\left(\sigma\left(\hat{y}_{t+1} - \hat{y}_t\right) - z_{t+1}^b + \frac{\omega_u c^u}{y}\frac{\zeta}{1+\zeta}\rho_\zeta \widehat{b}_{L,t} + \hat{\pi}_{t+1}\right)\right. \\
&\left. + \frac{\omega_u c^u}{y}\hat{\imath}_t - \hat{\pi}_{t+1} + z_{t+1}^b\right] \\
&= E_t\left[-\frac{\omega_u c^u \sigma}{y}\left(\hat{y}_{t+1} - \hat{y}_t\right) + \frac{\omega_u c^u}{y}\hat{\imath}_t - \frac{\omega_u c^u}{y}\left(\hat{\pi}_{t+1} - z_{t+1}^b\right) + \frac{(1-\omega_u)\,c^r}{y}\frac{\omega_u c^u}{y}\frac{\zeta}{1+\zeta}\rho_\zeta \widehat{b}_{L,t}\right],
\end{aligned}$$

or

$$0 = E_t \left[-\sigma \left(\hat{y}_{t+1} - \hat{y}_t \right) + \hat{\imath}_t - \hat{\pi}_{t+1} + z^b_{t+1} + \frac{\left(1-\omega_u\right) c^r}{y} \frac{\zeta}{1+\zeta} \rho_\zeta \widehat{b}_{L,t} \right],$$

or

$$\begin{aligned}
\hat{y}_t &= E_t \hat{y}_{t+1} - \frac{1}{\sigma} \left(\hat{\imath}_t - E_t \hat{\pi}_{t+1} + E_t z^b_{t+1} \right) - \frac{1}{\sigma} \frac{\left(1-\omega_u\right) c^r}{y} \frac{\zeta}{1+\zeta} \rho_\zeta \widehat{b}_{L,t} \\
&= E_t \hat{y}_{t+1} - \frac{1}{\sigma} \left(\hat{\imath}_t - E_t \hat{\pi}_{t+1} \right) - \frac{1}{\sigma} \frac{\left(1-\omega_u\right) c^r}{y} \frac{\zeta}{1+\zeta} \rho_\zeta \widehat{b}_{L,t} - \frac{\rho_b}{\sigma} z^b_t.
\end{aligned}$$

This equation shows that the central bank's government bond purchase – a decrease in $\hat{b}_{L,t}$ – stimulates output, given $E_t \hat{y}_{t+1}$ and the real rate $\hat{\imath}_t - E_t \hat{\pi}_{t+1}$. Since $\hat{b}_{L,t}$ follows the simple rule (A.17), the equation can be written as equation (9), where

$$\chi_b = \frac{\rho_b}{\sigma} > 0, \tag{A.29}$$

$$\lambda^* = \frac{\left(1-\omega_u\right) c^r}{y} \frac{\zeta}{1+\zeta} \rho_\zeta \gamma. \tag{A.30}$$

The case of $\lambda^* = 1$ corresponds to the fully effective UMP, which makes the ELB irrelevant. Such a case can be achieved, e.g. when the central bank responds to the shadow rate aggressively enough to satisfy:

$$\gamma = \left[\frac{\left(1-\omega_u\right) c^r}{y} \frac{\zeta}{1+\zeta} \rho_\zeta \right]^{-1}.$$

Phillips curve The Phillips curve can be derived from equations (A.10)-(A.13). Log-linearizing equation (A.13) yields:

$$\widehat{\tilde{p}}_t = -\frac{\xi}{1-\xi} \widehat{\tilde{\Pi}}^p_{t|t-1}, \tag{A.31}$$

where

$$\widehat{\tilde{\Pi}}^p_{t|t-1} = \left(1-\nu_p\right) \hat{\pi}_{t-1} - \hat{\pi}_t.$$

Loglinearizing equation (A.10) yields:

$$\begin{aligned}
\frac{\theta-\lambda_p}{\left(1-\lambda_p\right)\theta} \widehat{\tilde{p}}_t =& \frac{\omega_u K^u_p}{\omega_u K^u_p + \left(1-\omega_u\right) K^r_p} \hat{K}^u_{p,t} + \frac{\left(1-\omega_u\right) K^r_p}{\omega_u K^u_p + \left(1-\omega_u\right) K^r_p} \hat{K}^r_{p,t} \\
&- \frac{\omega_u F^u_p}{\omega_u F^u_p + \left(1-\omega_u\right) F^r_p} \hat{F}^u_{p,t} - \frac{\left(1-\omega_u\right) F^r_p}{\omega_u F^u_p + \left(1-\omega_u\right) F^r_p} \hat{F}^r_{p,t}.
\end{aligned} \tag{A.32}$$

Combining equations (A.31) and (A.32) leads to:

$$\begin{aligned}
-\frac{\xi}{1-\xi} \frac{\theta-\lambda_p}{\left(1-\lambda_p\right)\theta} \left[\left(1-\nu_p\right) \hat{\pi}_{t-1} - \hat{\pi}_t \right] =& \frac{\omega_u K^u_p}{\omega_u K^u_p + \left(1-\omega_u\right) K^r_p} \hat{K}^u_{p,t} + \frac{\left(1-\omega_u\right) K^r_p}{\omega_u K^u_p + \left(1-\omega_u\right) K^r_p} \hat{K}^r_{p,t} \\
&- \frac{\omega_u F^u_p}{\omega_u F^u_p + \left(1-\omega_u\right) F^r_p} \hat{F}^u_{p,t} - \frac{\left(1-\omega_u\right) F^r_p}{\omega_u F^u_p + \left(1-\omega_u\right) F^r_p} \hat{F}^r_{p,t}.
\end{aligned} \tag{A.33}$$

Log-linearizing equation (A.11) and (A.12) yields:

$$\begin{aligned}
\hat{F}^j_{p,t} &= (1-\xi\delta) \left(-\sigma \hat{c}^j_t + \hat{y}_t \right) + \xi\delta E_t \left(z^b_{t+1} + \frac{1}{1-\lambda_p} \widehat{\tilde{\Pi}}^p_{t+1|t} + \hat{F}^j_{p,t+1} \right), \\
\hat{K}^j_{p,t} &= (1-\xi\delta) \left(-\sigma \hat{c}^j_t + \frac{1}{\theta} \hat{y}_t - \frac{1}{\theta} z^a_t + \hat{w}_t \right) + \xi\delta E_t \left(z^b_{t+1} + \frac{\lambda_p}{\left(1-\lambda_p\right)\theta} \widehat{\tilde{\Pi}}^p_{t+1|t} + \hat{K}^j_{p,t+1} \right),
\end{aligned}$$

for $j = r$ and u. The term involving $\hat{F}^u_{p,t}$ and $\hat{F}^r_{p,t}$ in equation (A.33) is calculated as follows.

$$\begin{aligned}
&\frac{\omega_u F^u_p}{\omega_u F^u_p + (1-\omega_u) F^r_p}\hat{F}^u_{p,t} + \frac{(1-\omega_u) F^r_p}{\omega_u F^u_p + (1-\omega_u) F^r_p}\hat{F}^r_{p,t} \\
&= (1-\xi\delta)\left(-\sigma\frac{\omega_u F^u_p \hat{c}^u_t + (1-\omega_u) F^r_p \hat{c}^r_t}{\omega_u F^u_p + (1-\omega_u) F^r_p} + \hat{y}_t\right) \\
&+ \xi\delta E_t\left(z^b_{t+1} + \frac{1}{1-\lambda_p}\widehat{\widetilde{\Pi}}^p_{t+1|t} + \frac{\omega_u F^u_p}{\omega_u F^u_p + (1-\omega_u) F^r_p}\hat{F}^u_{p,t+1} + \frac{(1-\omega_u) F^r_p}{\omega_u F^u_p + (1-\omega_u) F^r_p}\hat{F}^r_{p,t+1}\right).
\end{aligned}$$

Similarly, the term involving $\hat{K}^u_{p,t}$ and $\hat{K}^r_{p,t}$ in equation (A.33) is calculated as:

$$\begin{aligned}
&\frac{\omega_u K^u_p}{\omega_u K^u_p + (1-\omega_u) K^r_p}\hat{K}^u_{p,t} + \frac{(1-\omega_u) K^r_p}{\omega_u K^u_p + (1-\omega_u) K^r_p}\hat{K}^r_{p,t} \\
&= (1-\xi\delta)\left(-\sigma\frac{\omega_u K^u_p \hat{c}^u_t + (1-\omega_u) K^r_p \hat{c}^r_t}{\omega_u K^u_p + (1-\omega_u) K^r_p} + \frac{1}{\theta}\hat{y}_t - \frac{1}{\theta}z^a_t + \hat{w}_t\right) \\
&+ \xi\delta E_t\left(z^b_{t+1} + \frac{\lambda_p}{(1-\lambda_p)\theta}\widehat{\widetilde{\Pi}}^p_{t+1|t} + \frac{\omega_u K^u_p}{\omega_u K^u_p + (1-\omega_u) K^r_p}\hat{K}^u_{p,t+1} + \frac{(1-\omega_u) K^r_p}{\omega_u K^u_p + (1-\omega_u) K^r_p}\hat{K}^r_{p,t+1}\right).
\end{aligned}$$

Let the right-hand-side of equation (A.33) denote as $\hat{X}_t$. Then, using the above relationships just derived, $\hat{X}_t$ can be written as:

$$\hat{X}_t = (1-\xi\delta)\left[\left(\frac{1}{\theta}-1\right)\hat{y}_t - \frac{1}{\theta}z^a_t + \hat{w}_t\right] + \xi\delta E_t\left(-\frac{\lambda_p-\theta}{(\lambda_p-1)\theta}\widehat{\widetilde{\Pi}}^p_{t+1|t} + \hat{X}_{t+1}\right)$$

Because $\hat{X}_t$ is the right-hand-side of equation (A.33), equation (A.33) can be written as:

$$\begin{aligned}
-\frac{\xi}{1-\xi}\frac{\lambda_p-\theta}{(\lambda_p-1)\theta}\left[(1-\nu_p)\hat{\pi}_{t-1} - \hat{\pi}_t\right] &= (1-\xi\delta)\left[\left(\frac{1}{\theta}-1\right)\hat{y}_t - \frac{1}{\theta}z^a_t + \hat{w}_t\right] \\
&+ \xi\delta E_t\left(-\frac{\lambda_p-\theta}{(\lambda_p-1)\theta}\widehat{\widetilde{\Pi}}^p_{t+1|t} - \frac{\xi}{1-\xi}\frac{\lambda_p-\theta}{(\lambda_p-1)\theta}\left[(1-\nu_p)\hat{\pi}_t - \hat{\pi}_{t+1}\right]\right),
\end{aligned}$$

or

$$\hat{\pi}_t = \frac{\xi(1-\nu_p)}{(\xi+1-\nu_p)}\hat{\pi}_{t-1} + \frac{(1-\xi\delta)(1-\xi)(\lambda_p-1)\theta}{(\lambda_p-\theta)(\xi+1-\nu_p)}\left[\left(\frac{1}{\theta}-1\right)\hat{y}_t - \frac{1}{\theta}z^a_t + \hat{w}_t\right] + \frac{\xi\delta}{(\xi+1-\nu_p)}E_t\hat{\pi}_{t+1}.$$

From equations (A.4) and (A.7), the wage $\hat{w}_t$ can be written as:

$$\begin{aligned}
\hat{w}_t &= \omega_u\left(\sigma\hat{c}^u_t + \frac{1}{\nu}\hat{h}^u_t\right) + (1-\omega_u)\left(\sigma\hat{c}^r_t + \frac{1}{\nu}\hat{h}^r_t\right), \\
&= \sigma\hat{y}_t + \frac{1}{\nu}\hat{h}_t = \left(\sigma + \frac{1}{\nu\theta}\right)\hat{y}_t - \frac{1}{\nu\theta}z^a_t,
\end{aligned}$$

where the market clearing conditions (A.3) and (A.18) were used in the second equality and the production function (A.9) was used in the third equality. Since $c^u = c^r$ is assumed, the second equality holds. By using the expression for $\hat{w}_t$, the Phillips curve can be written as

$$\begin{aligned}
\hat{\pi}_t &= \frac{\xi(1-\nu_p)}{(\xi+1-\nu_p)}\hat{\pi}_{t-1} \\
&+ \frac{(1-\xi\delta)(1-\xi)(\lambda_p-1)\theta}{(\lambda_p-\theta)(\xi+1-\nu_p)}\left[\frac{\nu+\nu\theta(\sigma-1)+1}{\nu\theta}\hat{y}_t - \frac{1+\nu}{\nu\theta}z^a_t\right] + \frac{\xi\delta}{(\xi+1-\nu_p)}E_t\hat{\pi}_{t+1}.
\end{aligned}$$

In the case of no price indexation to the past inflation rate and a linear production function, that is, in the case of $\nu_p = 1$ and $\theta = 1$, the Phillips curve is collapsed to the standard form:

$$\hat{\pi}_t = \frac{(1-\xi\delta)(1-\xi)}{\xi}\left(\sigma + \frac{1}{\nu}\right)\hat{y}_t + \delta E_t\hat{\pi}_{t+1} - \frac{(1-\xi\delta)(1-\xi)}{\xi}\frac{1+\nu}{\nu}z^a_t.$$

This completes the derivation of equation (10) where

$$\kappa = \frac{(1-\xi\delta)(1-\xi)}{\xi}\left(\sigma + \frac{1}{\nu}\right), \tag{A.34}$$

$$\chi_a = \frac{(1-\xi\delta)(1-\xi)}{\xi}\frac{1+\nu}{\nu}. \tag{A.35}$$

A.3 Parameterization of the model

Instead of parameterizing the model presented in Appendix A.1, we parameterize the system of log-linearized equations (9)-(13). It is worth emphasizing that the parameterized model is used for illustrating the implications of the theoretical model, and not for deriving quantitative implications, which would require a more complex system.

The relative risk aversion σ is set at $\sigma = 2$. The discount factor is set close to unity at $\delta = 0.997$. The slope of the Phillips curve κ is set at $\kappa = 0.336$ using equation (A.34) with the Calvo parameter of $\xi = 0.75$ and the Frisch labor elasticity of $\nu = 0.5$. In the monetary policy rule, the persistence parameter is set at $\rho_i = 0.7$; the inflation coefficient is set at $r_\pi = 1.5$; the output coefficient is set at $r_y = 0.5$. The AR(1) coefficients for the productivity and preference shocks are set at $\rho_a = \rho_b = 0.9$, and the coefficients χ_b and χ_a are set according to equations (A.29) and (A.35), respectively. We set UMP parameters, λ^* and α, freely to numerically examine the effects of UMP.

A.4 Proof of Proposition 1

Part (i) Because of the equivalence established in Lemma 1, without loss of generality, consider the case of $\lambda^* = 1$ and $\alpha = 0$ in the theoretical model. In this case, the variables $\hat{y}_t$, $\hat{\pi}_t$, and $\hat{i}^*_t$ have a closed system of equations, consisting of equation (9) with $\lambda^* = 1$, equation (10), and $\hat{i}^*_t = \hat{i}^{\text{Taylor}}_t$, where $\hat{i}^{\text{Taylor}}_t$ is given by equation (12).

In this case, the state of the economy in period t can be summarized by $\hat{i}^*_{t-1}$, ϵ^i_t, z^a_t, and z^b_t. Then decision rules for $\hat{y}_t$ and $\hat{\pi}_t$ have the following form:

$$\begin{aligned}
\hat{y}_t &= d_{yi^*}\hat{i}^*_{t-1} + d_{yi}\epsilon^i_t + d_{ya}z^a_t + d_{yb}z^b_t,\\
\hat{\pi}_t &= d_{\pi i^*}\hat{i}^*_{t-1} + d_{\pi i}\epsilon^i_t + d_{\pi a}z^a_t + d_{\pi b}z^b_t,
\end{aligned}$$

with coefficients $\{d_{yi^*}, d_{yi}, d_{ya}, d_{yb}, d_{\pi i^*}, d_{\pi i}, d_{\pi a}, d_{\pi b}\}$ uniquely determined under standard assumptions of the model (such as the Taylor principle). With these decision rules, the equation for $\hat{i}^*_t$ can be written as

$$\begin{aligned}
\hat{i}^*_t =& \left[\rho_i + (1-\rho_i)(r_\pi d_{\pi i^*} + r_y d_{yi^*})\right]\hat{i}^*_{t-1} + \left[(1-\rho_i)(r_\pi d_{\pi i} + r_y d_{yi}) + 1\right]\epsilon^i_t\\
&+(1-\rho_i)(r_\pi d_{\pi a} + r_y d_{ya})z^a_t + (1-\rho_i)(r_\pi d_{\pi b} + r_y d_{yb})z^b_t\\
=& d_{i^*i^*}\hat{i}^*_{t-1} + d_{i^*i}\epsilon^i_t + d_{i^*a}z^a_t + d_{i^*b}z^b_t.
\end{aligned}$$

Let $\mathbf{y}_t \equiv [\hat{y}_t, \hat{\pi}_t, \hat{i}^*_t]'$ denote the vector of endogenous variables. The decision rule implies

$$\begin{aligned}
\mathbf{y}_t =& \begin{bmatrix} d_{yi^*} & d_{yi} & d_{ya} & d_{yb}\\ d_{\pi i^*} & d_{\pi i} & d_{\pi a} & d_{\pi b}\\ d_{i^*i^*} & d_{i^*i} & d_{i^*a} & d_{i^*b}\end{bmatrix}\begin{bmatrix} i^*_{t-1}\\ \epsilon^i_t\\ \rho_a z^a_{t-1} + \epsilon^a_t\\ \rho_b z^b_{t-1} + \epsilon^b_t\end{bmatrix}\\
=& \begin{bmatrix} d_{yi^*} & \rho_a d_{ya} & \rho_b d_{yb}\\ d_{\pi i^*} & \rho_a d_{\pi a} & \rho_b d_{\pi b}\\ d_{i^*i^*} & \rho_a d_{i^*a} & \rho_b d_{i^*b}\end{bmatrix}\begin{bmatrix} i^*_{t-1}\\ z^a_{t-1}\\ z^b_{t-1}\end{bmatrix} + \begin{bmatrix} d_{yi} & d_{ya} & d_{yb}\\ d_{\pi i} & d_{\pi a} & d_{\pi b}\\ d_{i^*i} & d_{i^*a} & d_{i^*b}\end{bmatrix}\begin{bmatrix}\epsilon^i_t\\ \epsilon^a_t\\ \epsilon^b_t\end{bmatrix}\\
=& \mathbf{C}\mathbf{x}_{t-1} + \mathbf{D}\epsilon_t.
\end{aligned} \tag{A.36}$$

The law of motion for $\mathbf{x}_t \equiv [\hat{i}_t^*, z_t^a, z_t^b]'$ is:

$$\mathbf{x}_t = \begin{bmatrix} d_{i^*i^*} & \rho_a d_{i^*a} & \rho_b d_{i^*b} \\ 0 & \rho_a & 0 \\ 0 & 0 & \rho_b \end{bmatrix} \mathbf{x}_{t-1} + \begin{bmatrix} d_{i^*i} & d_{i^*a} & d_{i^*b} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \epsilon_t$$
$$= \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\epsilon_t. \tag{A.37}$$

Solving equation (A.36) for ϵ_t, and substituting the outcome in equation (A.37) yields:

$$\mathbf{x}_t = \left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)\mathbf{x}_{t-1} + \mathbf{B}\mathbf{D}^{-1}\mathbf{y}_t.$$

If $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \mathbf{0}$, the vector of endogenous variables, $\mathbf{y}_t$, has a VAR(1) representation:

$$\mathbf{y}_t = \mathbf{C}\mathbf{B}\mathbf{D}^{-1}\mathbf{y}_{t-1} + D\epsilon_t.$$

The rest of the proof shows $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \mathbf{0}$. Substituting the matrices $\mathbf{A}$ and $\mathbf{B}$ in equation (A.37) into this condition yields:

$$\mathbf{D}^{-1}\mathbf{C} = \begin{bmatrix} d_{i^*i^*}/d_{i^*i} & 0 & 0 \\ 0 & \rho_a & 0 \\ 0 & 0 & \rho_b \end{bmatrix}.$$

Further substituting the matrices $\mathbf{C}$ and $\mathbf{D}$ in equation (A.36) into this condition leads to: $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} = \mathbf{0}$ if and only if $d_{yi^*} = d_{yi}\left(d_{i^*i^*}/d_{i^*i}\right)$ and $d_{\pi i^*} = d_{\pi i}\left(d_{i^*i^*}/d_{i^*i}\right)$. Substituting the decision rules into equations (9) yields:

$$\hat{y}_t = \left(d_{yi^*} - \frac{1}{\sigma} + \frac{d_{\pi i}}{\sigma}\right) d_{i^*i^*}\hat{i}_{t-1}^* + \left(d_{yi^*} - \frac{1}{\sigma} + \frac{d_{\pi i}}{\sigma}\right) d_{i^*i}\epsilon_t^i + ...,$$

where terms related to z_t^a and z_t^b are omitted. Matching coefficients on $\hat{i}_{t-1}^*$ and ϵ_t^i of both sides of the equation yields:

$$d_{yi^*} = \left(d_{yi^*} - \frac{1}{\sigma} + \frac{d_{\pi i}}{\sigma}\right) d_{i^*i^*},$$
$$d_{yi} = \left(d_{yi^*} - \frac{1}{\sigma} + \frac{d_{\pi i}}{\sigma}\right) d_{i^*i}.$$

These two equations imply $d_{yi^*} = d_{yi}\left(d_{i^*i^*}/d_{i^*i}\right)$. Next, substituting the decision rules into equation (10) yields:

$$\hat{\pi}_t = \left(\delta d_{\pi i^*} + \kappa d_{yi^*}\right)\hat{i}_{t-1}^* + \left(\delta d_{\pi i^*}d_{i^*i} + \kappa d_{yi}\right)\epsilon_t^i + ...,$$

where terms related to z_t^a and z_t^b are omitted. Matching coefficients on $\hat{i}_{t-1}^*$ and ϵ_t^i of both sides of the equation yields:

$$d_{\pi i^*} = \delta d_{\pi i^*} + \kappa d_{yi}\left(\frac{d_{i^*i^*}}{d_{i^*i}}\right),$$
$$d_{\pi i} = \delta d_{\pi i^*}d_{i^*i} + \kappa d_{yi},$$

where $d_{yi^*} = d_{yi}\left(d_{i^*i^*}/d_{i^*i}\right)$ is used in the first equation. Solving these two equations for $d_{\pi i^*}$ yields $d_{\pi i^*} = d_{\pi i}\left(d_{i^*i^*}/d_{i^*i}\right)$.

Part (ii) Again, without loss of generality, consider the case of $\lambda^* = 1$ and $\alpha = 0$. From equations (9) and (A.27), the return of the long-term government bond and the shadow rate are linked as follows:

$$\frac{(1-\omega_u)c^r}{y} E_t\hat{R}_{L,t+1} + \frac{\omega_u c^u}{y}\hat{i}_t = (1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*.$$

When $\lambda^* = 1$, this equation implies $E_t\hat{R}_{L,t+1} = \hat{i}_t^*$, which can be rewritten by using $R_{L,t+1} = \bar{R}_{L,t+1}(\bar{R}_{L,t} - \mu)/(\bar{R}_{L,t+1} - \mu)$ as

$$\hat{\bar{R}}_{L,t} = \frac{\bar{R}_L - \mu}{\bar{R}_L}\hat{i}_t^* + \frac{\mu}{\bar{R}_L}E_t\hat{\bar{R}}_{L,t+1},$$

where $\bar{R}_L < \mu$ in steady state. Solving this equation forward yields

$$\hat{\hat{R}}_{L,t} = \left(\frac{\bar{R}_L - \mu}{\bar{R}_L}\right) E_t \left[\hat{i}_t^* + \frac{\mu}{\bar{R}_L}\hat{i}_{t+1}^* + \left(\frac{\mu}{\bar{R}_L}\right)^2 \hat{i}_{t+2}^* + ...\right].$$

Because the right-hand-side of the equation depends on inflation in period t, that is $\hat{i}_t^*$, z_t^a, and z_t^b, the long-term interest rate can be written as:

$$\hat{\hat{R}}_{L,t} = f_{i^*}\hat{i}_t^* + f_a z_t^a + f_b z_t^b,$$

where f_{i^*}, f_a, and f_b are coefficients. By using the decision rule for the shadow rate, this equation can be written as:

$$\hat{\hat{R}}_{L,t} = f_{i^*}d_{i^*i^*}\hat{i}_{t-1}^* + f_{i^*}d_{i^*i}\epsilon_t^i + (f_{i^*}d_{i^*a} + f_a)\, z_t^a + (f_{i^*}d_{i^*b} + f_b)\, z_t^b.$$

Define $\tilde{\mathbf{y}}_t \equiv [\hat{y}_t, \hat{\pi}_t, \hat{\hat{R}}_{L,t}]'$. Then, the state space representation for $\tilde{\mathbf{y}}_t$ is

$$\begin{aligned}\tilde{\mathbf{y}}_t =& \begin{bmatrix} d_{yi^*} & \rho_a d_{ya} & \rho_b d_{yb} \\ d_{\pi i^*} & \rho_a d_{\pi a} & \rho_b d_{\pi b} \\ f_{i^*}d_{i^*i^*} & \rho_a(f_{i^*}d_{i^*a} + f_a) & \rho_b(f_{i^*}d_{i^*b} + f_b) \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} d_{yi} & d_{ya} & d_{yb} \\ d_{\pi i} & d_{\pi a} & d_{\pi b} \\ f_{i^*}d_{i^*i} & f_{i^*}d_{i^*a} + f_a & f_{i^*}d_{i^*b} + f_b \end{bmatrix} \epsilon_t \\ =& \tilde{\mathbf{C}}\mathbf{x}_{t-1} + \tilde{\mathbf{D}}\epsilon_t.\end{aligned}$$

Similar to the part (i) in Proposition 1, a solution for $\tilde{\mathbf{y}}_t$ can have a VAR(1) representation if and only if $\mathbf{A} - \mathbf{B}\tilde{\mathbf{D}}^{-1}\tilde{\mathbf{C}} = \mathbf{0}$. This condition holds if and only if $d_{yi^*} = d_{yi}(d_{i^*i^*}/d_{i^*i})$ and $d_{\pi i^*} = d_{\pi i}(d_{i^*i^*}/d_{i^*i})$. The latter two conditions hold as shown in Part (i).

A.5 Proof of Proposition 2

Without loss of generality, consider the case of agents forming expectations assuming: $\lambda^* = 1$ and $\alpha = 0$. When forming expectations about variables in period $t+1$, the initial condition is given by $\tilde{\mathbf{x}}_t \equiv [(1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*, z_t^a, z_t^b]'$. Under this assumption about expectations, the decision rule used for forming expectations about period $t+1$ variables is $\mathbf{y}_{t+1} = \mathbf{C}\tilde{\mathbf{x}}_t + \mathbf{D}\epsilon_{t+1}$. From period $t+s$ onward, for $s = 2, 3, ...$, time $t+s$ variables are expected *in period* t to follow $\mathbf{y}_{t+s} = \mathbf{C}\mathbf{x}_{t+s-1} + \mathbf{D}\epsilon_{t+s}$. But, once the time proceeds and becomes period $t+1$, the initial condition is updated to $\tilde{\mathbf{x}}_{t+1}$ and this is used for forming expectations about $t+2$ variables as $E_{t+1}\mathbf{y}_{t+2} = \mathbf{C}\tilde{\mathbf{x}}_{t+1}$. Hence, under the assumption about expectations, the decision rule is given by $\mathbf{y}_{t+s} = \mathbf{C}\tilde{\mathbf{x}}_{t+s-1} + \mathbf{D}\epsilon_{t+s}$ for $s = 1, 2, ...$ In this system, in every period information is updated and $\tilde{\mathbf{x}}_{t+s-1}$ is used as an initial condition. The interest rate $\hat{i}_{t+s-1}$ in the initial condition is treated as if it were an exogenous variable.

Substituting the decision rule into equations (9) and (10) yields:

$$\begin{aligned}\hat{y}_t =& \left(-\frac{1}{\sigma} + d_{yi^*} + \frac{d_{\pi i^*}}{\sigma}\right)\left((1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*\right) + \\ &+ \left(\rho_a d_{ya} + \frac{\rho_a d_{\pi a}}{\sigma}\right) z_t^a + \left(\rho_b d_{yb} + \frac{\rho_b d_{\pi b}}{\sigma} - \chi_z\right) z_t^b, \qquad (A.38) \\ -\kappa y_t + \hat{\pi}_t =& \delta d_{\pi i^*}\left((1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*\right) + (\delta d_{\pi a} - \chi_a) z_t^a + \delta d_{\pi b} z_t^b. \qquad (A.39)\end{aligned}$$

Since z_t^a and z_t^b follow AR(1) processes, equations (A.38) and (A.39) these two equations can be written in a matrix form as:

$$\mathbf{H}_1 \begin{bmatrix}\hat{y}_t \\ \hat{\pi}_t\end{bmatrix} = \mathbf{H}_2\left((1-\lambda^*)\hat{i} + \lambda^*\hat{i}_t^*\right) + \mathbf{H}_3\begin{bmatrix} z_{t-1}^a \\ z_{t-1}^b\end{bmatrix} + \mathbf{H}_4\begin{bmatrix}\epsilon_t^a \\ \epsilon_t^b\end{bmatrix}.$$

or

$$\begin{bmatrix}\hat{y}_t \\ \hat{\pi}_t\end{bmatrix} = \mathbf{H}_1^{-1}\mathbf{H}_2\left((1-\lambda^*)\hat{i} + \lambda^*\hat{i}_t^*\right) + \mathbf{H}_1^{-1}\mathbf{H}_3\begin{bmatrix} z_{t-1}^a \\ z_{t-1}^b\end{bmatrix} + \mathbf{H}_1^{-1}\mathbf{H}_4\begin{bmatrix}\epsilon_t^a \\ \epsilon_t^b\end{bmatrix}. \qquad (A.40)$$

Also, under the assumption about expectations, the expected values can be written as: $E_t\tilde{\mathbf{y}}_{t+1} = \mathbf{G}\tilde{\mathbf{y}}_t$, where $\tilde{\mathbf{y}}_t \equiv [\hat{y}_t, \hat{\pi}_t, (1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*]'$ and $\mathbf{G} \equiv \mathbf{CBD}^{-1}$, as derived in the proof of Proposition 1. By using this equation, equations (9) and (10) can be written as:

$$\chi_z z_t^b = \left(g_{yy} + \frac{g_{\pi y}}{\sigma} - 1\right)\hat{y}_t + \left(g_{y\pi} + \frac{g_{\pi\pi}}{\sigma}\right)\hat{\pi}_t + \left(g_{yi^*} + \frac{g_{\pi i^*}}{\sigma} - \frac{1}{\sigma}\right)\left((1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*\right),$$
$$\chi_a z_t^a = (\delta g_{\pi y} + \kappa)\hat{y}_t + (\delta g_{\pi\pi} - 1)\hat{\pi}_t + \delta g_{\pi i^*}\left((1-\lambda^*)\hat{i}_t + \lambda^*\hat{i}_t^*\right),$$

where g_{ij}'s correspond to elements in the matrix $\mathbf{G}$. Then, the lagged shocks z_{t-1}^b and z_{t-1}^a in equation (A.40) can be represented by a function of $\tilde{\mathbf{y}}_{t-1} \equiv [\hat{y}_{t-1}, \hat{\pi}_{t-1}, (1-\lambda^*)\hat{i}_{t-1} + \lambda^*\hat{i}_{t-1}^*]'$. From this result, equation (A.40) is in the same form of equation (1) in the structural VAR.

B Data description

We construct our quarterly data by taking averages of monthly series. For the U.S., the inflation rate is computed from the implicit price deflator (GDPDEF) as $\pi_t = 400 \times \log(P_t/P_{t-1})$, where P_t is the GDP deflator. The output gap is calculated as $100\% \times (GDPC1 - GDPPOT)/GDPPOT$, where GDPC1 is the series for the U.S. real GDP and GDPPOT is the U.S. real potential GDP. The long-term interest rate is from the 10-year Treasury constant maturity rate (GS10). All these series are from the FRED database.[33] Money growth data for the U.S. are computed from 12 alternative indicators as listed in Table 4 as $m_t = 400\times\log(M_t/M_{t-1})$, where M_t is the particular money series considered. All M_t values are quarterly and computed by taking averages of their corresponding monthly values. The traditional monetary aggregates (MB, M1, M2, M2M, MZM), and securities held outright are from the FRED database. The Divisia monetary aggregates (DIVM1, DIVM2, DIVM2M, DIVMZM, DIVM4) are from the Center for Financial Stability Divisia database.

For Japan, the quarterly Call Rate, bond yields, and the core CPI are computed as the averages of their monthly counterparts. The quarterly the inflation rate is computed from the core CPI (consumption tax changes adjusted) as $\pi_t = 400 \times (CPI_t - CPI_{t-1})/CPI_{t-1}$. The GDP gap is that published by the Bank of Japan. The trend growth is defined by the annualised growth rate of potential GDP from the previous quarter, which comes from the estimates of the Cabinet Office. The interest on reserves (IOR) is constructed from the interest rate that the BoJ applies to the Complementary Deposit Facility. [34]

[33] The data can be retrieved from the following websites: GDP deflator https://fred.stlouisfed.org/series/GDPDEF; and series to construct the output gap: https://fred.stlouisfed.org/series/GDPC1 and https://fred.stlouisfed.org/series/GDPPOT; the Federal Funds Rate https://fred.stlouisfed.org/series/FEDFUNDS; and the long yield https://fred.stlouisfed.org/series/GS10. The treatment of the data is described in Appendix B. The data for the different monetary aggregates is available at: https://fred.stlouisfed.org/categories/24 and http://www.centerforfinancialstability.org/amfm_data.php.

[34] The data can be retrieved from the following websites: Call Rate (Bank of Japan): http://www.stat-search.boj.or.jp/index_en.html; 9-year and 10-year government bond yields (Ministry of Finance): https://www.mof.go.jp/jgbs/reference/interest_rate/data/jgbcm_all.csv; GDP gap (Bank of Japan): https://www.boj.or.jp/en/research/research_data/index.htm/; core CPI inflation (Statistics Bureau of Japan): https://www.e-stat.go.jp/stat-search/file-download?statInfId=000031431696&fileKind=1; trend growth rate (Cabinet Office): https://www5.cao.go.jp/keizai3/getsurei-e/index-e.html.

Table 4: Monetary Aggregates Data used in the Model

Monetary Aggregate (M_t)	Mnemonics in the Corresponding Database	Available Sample Periods
Monetary Base (MB)	MBSL	1948Q1-2019Q1
M1	M1SL	1959Q2-2019Q1
M2	M2SL	1959Q2-2019Q1
M2M	M2MSL	1959Q2-2019Q1
MZM	MZMSL	1959Q2-2019Q1
Securities Held Outright	WSECOUT	1989Q3-2019Q1
Divisia M1 (DIVM1)	Divisia M1	1967Q2-2019Q1
Divisia M2 (DIVM2)	Divisia M2	1967Q2-2019Q1
Divisia M2M (DIVM2M)	Divisia M2M	1967Q2-2019Q1
Divisia MZM (DIVMZM)	Divisia MZM	1967Q2-2019Q1
Divisia M4 (DIVM4)	DM4	1967Q2-2019Q1

C Additional empirical results

Table 5 shows the results of tests for exclusion of the Federal Funds Rate from an SVAR that includes inflation, the output gap, the 10-year bond yield, and various alternative measures of the growth of money outlined in column (1). Column (3) shows the order of the VAR selected by the AIC, which varies between 3 and 4 lags, consistent with the benchmark model in Table 1. Columns (4) and (6) report the likelihood ratio test statistics for the joint exclusion hypothesis and the corresponding asymptotic p-values, respectively. These results show that the data strongly and consistently reject the joint exclusion restrictions on the Federal Funds Rate across all the alternative specifications for all measures of money supply, which corroborates the findings in the baseline 4-equation model in Table 1.

Robustness of test results for the U.S. to great moderation The test results of the IH over the full sample are subject to a possible misspecification arising from the 'Great Moderation', a drop in U.S. macroeconomic volatility in the mid-1980s. Therefore, we assess the robustness of our results by estimating the model and performing the above tests of the IH over the sub-sample which starts in 1984q1. Tables 6 and 7 report the results over this subsample, which correspond to the results reported in Tables 1 and 2 for the full sample, respectively. The results of the tests of the IH remain the same: the hypothesis is firmly rejected.

Robustness of Japanese results to 10-year rates Similarly, we test the robustness of our results for the Japanese data by using the 10-year yields in the model instead. This shortens the available sample for estimation to 1987q4 to 2019q1. Tables 8 - 9 report test statistics for the 3 types of tests for the IH. From Tables 8 and 9, the IH is rejected across all lags. For the CKSVAR alternative, we select 2 lags based in the Akaike criterion. Then Table 9 also suggests the rejection of the IH.

Tests of excluding long rates from VAR Table 10 shows results of this test for the U.S. and Japan in the KSVAR specification. The AIC reported in the table indicates that the number of lags that best fit the data is 3 for the U.S. and 2 for Japan, and the p-values reported in the table show that the null hypothesis of excluding long-term yields cannot be rejected in the SVAR with the preferred lags specification for the U.S. and Japan.

Table 5: Test for excluding short rates from VARs that include long rates and money

Mon. Aggr.	sample	p	LR	df	p-val
MB	1960q1–2019q1	3	52.16	16	0.0000
M1	1960q3-2019q1	3	53.79	16	0.0000
M2	1960q3-2019q1	3	53.51	16	0.0000
M2M	1960q3-2019q1	4	72.72	20	0.0000
MZM	1960q3-2019q1	4	78.02	20	0.0000
DIVM1	1968q3-2019q1	4	81.49	20	0.0000
DIVM2	1968q3-2019q1	4	113.32	20	0.0000
DIVM2M	1968q3-2019q1	4	112.16	20	0.0000
DIVMZM	1968q3-2019q1	4	107.22	20	0.0000
DIVM4	1968q3-2019q1	4	137.92	20	0.0000
SHO	1990q4-2019q1	3	93.32	16	0.0000

Note: The estimated model is a KSVAR(p) for the U.S. with inflation, output gap, the Federal Funds Rate, the 10-year government bond yield, and a different measure of money growth in each row. Sample availability varies for each monetary aggregate used. LR is the value of the LR test statistic for the testing that lags of the Federal Funds Rate can be excluded from all other equations in the model, df is number of exclusion restrictions, and p-value is the asymptotic χ^2_{df} p-value of the test.

Table 6: Test for excluding short rates form VAR that includes long rates post-1984

	KSVAR(p)						CKSVAR(p)					
p	loglik	pv-p	AIC	LR	df	p-val	loglik	pv-p	AIC	LR	df	p-val
5	103.42	-	-0.09	27.78	18	0.066	129.4	-	-0.18	70.71	33	0.000
4	97.22	0.715	-0.23	29.22	15	0.015	127.3	1.000	-0.43	81.61	27	0.000
3	88.25	0.550	-0.33	23.65	12	0.023	112.5	0.747	-0.50	62.78	21	0.000
2	67.25	0.013	-0.26	27.21	9	0.001	81.3	0.002	-0.35	46.76	15	0.000
1	20.80	0.000	0.17	9.94	6	0.127	27.7	0.000	0.13	20.75	9	0.014

Note: The estimated model is a (C)KSVAR(p) for the U.S. with inflation, output gap, Federal Funds Rate, and the 10-year government bond yield. Estimation sample is 1984q1-2019q1. loglik is the value of the log-likelihood. pv-p is the p-value of the test for lag reduction. AIC is the Akaike information criterion. LR test statistic for excluding short rates from equations for inflation, output gap and long rates. df is number of restrictions. Asymptotic p-values.

Table 7: Testing CSVAR against CKSVAR post-1984

Country	p	LR	df	p-val
U.S.	3	42.62	15	0.000

Note: The unrestricted model is a CKSVAR(3) for the U.S. with inflation, output gap, 10-year government bond yields, and the Federal Funds Rate. Sample: 1984q1-2019q1. LR test statistics of the restrictions that the model reduces to CSVAR(3). Lag order chosen by AIC. df is number of restrictions, asymptotic p-value reported.

Table 8: Test for excluding short rates from VAR for Japan using 10-year bond yields

	KSVAR(p)						CKSVAR(p)					
p	loglik	pv-p	AIC	LR	df	p-val	loglik	pv-p	AIC	LR	df	p-val
5	285.1	-	-2.92	37.43	18	0.005	320.5	-	-3.17	99.85	33	0.000
4	275.1	0.217	-3.02	31.62	15	0.007	307.4	0.159	-3.28	86.95	27	0.000
3	270.5	0.605	-3.20	32.05	12	0.001	290.6	0.023	-3.33	62.07	21	0.000
2	256.2	0.155	-3.23	24.60	9	0.003	274.3	0.004	-3.39	50.90	15	0.000
1	196.4	0.000	-2.53	22.84	6	0.001	212.8	0.000	-2.73	38.81	9	0.000

Note: The estimated model is a (C)KSVAR(p) for Japan with inflation, output gap, 10-year government bond yields, and the Call Rate. Estimation sample is 1987q4-2019q1. loglik is the value of the log-likelihood. pv-p is the p-value of the test for lag reduction. AIC is the Akaike information criterion. LR test statistic for excluding short rates from equations for inflation, output gap and long rates. df is number of restrictions. Asymptotic p-values reported.

Table 9: Testing CSVAR against CKSVAR for Japan using 10-year bond yields

Country	p	LR	df	p-val
Japan	2	47.54	11	0.000

Note: The unrestricted model is a CKSVAR(2) for Japan with inflation, output gap, 10-year government bond yields, and the BoJ Call Rate. Estimation sample: 1987q4-2019q1. LR test statistics of the restrictions that the model reduces to CSVAR(2). Lag order chosen by AIC. df is number of restrictions, asymptotic p-value reported.

Table 10: Test for excluding long rates from KSVAR that includes short rates

	U.S.						Japan					
p	loglik	pv-p	AIC	LR	df	*p*-val	loglik	pv-p	AIC	LR	df	*p*-val
5	-210.8	-	2.597	9.632	10	0.473	248.1	-	-2.180	10.272	10	0.417
4	-220.0	0.295	2.540	7.743	8	0.459	239.9	0.425	-2.295	8.665	8	0.371
3	-232.9	0.072	2.514	5.549	6	0.476	232.2	0.471	-2.417	8.372	6	0.212
2	-264.9	0.000	2.649	12.135	4	0.016	223.8	0.445	-2.530	6.307	4	0.177
1	-295.1	0.000	2.769	15.842	2	0.000	184.8	0.000	-2.190	10.485	2	0.005

Note: Estimated model is KSVAR(p) with inflation, output gap, long rate and policy rate. Estimation sample is 1960q1-2019q1 for the U.S. and 1985q3-2019q1 for Japan. Long rates are 10-year government bond yields for the U.S. and 9-year yields for Japan. loglik is the value of the log-likelihood. pv-p is the *p*-value of the test for lag reduction. AIC is the Akaike information criterion. LR test statistic for excluding long rates from equations for inflation, output gap and policy rate. df is number of restrictions. Asymptotic *p*-values reported.